ERGODIC THEORETICAL APPROACH TO INVESTIGATE MEMORY PROPERTIES OF HEAVY TAILED PROCESSES

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ERGODIC THEORETICAL APPROACH TO INVESTIGATE MEMORY PROPERTIES OF HEAVY TAILED PROCESSES

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A class of infinitely divisible processes includes not only well-known Lévy processes, but also a wide variety of processes such as the Gaussian and the stable processes, moving averages driven by Lévy processes (e.g., Ornstein-Uhlenbeck processes), and harmonizable processes. This dissertation focuses on the limit theorems for heavy tailed stationary infinitely divisible processes of a certain integral form.

In the light of a recently developed, ergodic theoretical approach, an infinitely divisible process with integral representation can be decomposed into two processes; one with short range dependence and the other with long range dependence. In the language of ergodic theory, the former process is generated by a dissipative flow, while the latter one is generated by a conservative flow. If the underlying flow is dissipative, the process is known to be identical to moving averages. On the other hand, only few attempts have been made on the study of the processes generated by conservative flows, and this dissertation discusses three limit theorems for such processes. Specifically, we establish the functional central limit theorem, the limit theorem on the sample autocovariance, and the functional limit theorem on the partial maxima.

Taking advantage of some ergodic theoretical notions, called pointwise dual ergodicity, the memory length in the process whose underlying flow is conservative can be quantified by a single parameter. It then turns out that the growth rates of partial sums, autocovariances, and partial maxima, together with the properties of their weak limits, all depend on not only heaviness of the marginal tail but also the memory length. In particular, the limiting process in the functional central limit theorem constitutes a new class of stable process. Similarly, a new class of Fréchet process can be derived

as weak limits for the normalized partial maxima. These new classes of weak limits exhibit dramatically different features that have never been observed in the limiting processes for moving averages.

Subsequently, we also propose a new notion, called a tail measure, as an infinite-dimensional object that can measure the dependence of extremes of stochastic processes or random fields with regularly varying tails. Focusing on stationary infinitely divisible processes of integral forms, we will investigate the connection between the ergodic theoretical properties of tail measures and those of the probability laws of the processes.

BIOGRAPHICAL SKETCH

Takashi Owada was born in early July of 1979 in Toyama, snowy countryside of Japan. Graduating from Toyama Chubu High School in 1998, he enrolled in Tokyo University. At first, he was interested in the study of science, particularly chemistry, but as time went by, his academic interests shifted from science to economics. After completing the bachelor's degree in economics in 2002, he joined Master of Statistics program in the same institute. Upon graduation in 2004, he started working for Bank of Japan (central bank of Japan) as an economist; however, he then realized that he feels more excited about doing academic research. Finally he decided to quit the bank, and in 2007, he came to Cornell to pursue his Ph.D. In August 2013, he received his Doctor of Philosophy degree in Operations Research with concentration in applied probability and statistics. He will join Electrical Engineering of Technion - Israel Institute of Technology as a postdoctoral researcher soon after completing his Ph.D.

This thesis is dedicated to Hitomi, and my mother Masako Owada.

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CHAPTER 1

INTRODUCTION

1.1 Heavy-Tail Analysis and Stable Distributions

Heavy-tail analysis typically assumes that a random variable X has an algebraically decaying tail:

$$P(X > x) \sim Cx^{-\alpha}L(x), \quad \text{as } x \to \infty\,, \tag{1.1}$$

where $C > 0$ and $\alpha > 0$ are constants, and L is a slowly varying function; that is, $L(tx)/L(x) \to 1$ as $x \to \infty$ for all $t > 0$. X is then said to have a regularly varying tail with index α. If $0 < \alpha < 2$, X has infinite variance, and if $0 < \alpha < 1$, even the mean of X becomes infinite. By comparison, for a standard normal random variable N, Mill's ratio yields

$$P(N > x) \sim \frac{1}{x\sqrt{2\pi}} e^{-x^2/2} \quad \text{as } x \to \infty\,.$$

Thus, N has an exponentially decaying tail, which is significantly lighter than that of (1.1). Heavy-tail analysis applies to the systems governed by a series of extremal events that occur at a non-negligible rate. Under such circumstances, using light-tail models and treating those extremal events as outliers can lead to a serious misunderstanding of the system, and (1.1) is presented as a more plausible alternative. Indeed, the heavy-tail assumption (1.1) has been applied to diverse fields such as data network analysis, finance, insurance, and natural disasters; for details, see Adler et al. (1998), Beirlant et al. (2004), de Haan and Ferreira (2006), Embrechts et al. (1997), and McNeil et al. (2005).

One of the most basic and useful distributions satisfying (1.1) is stable distribution. A random variable X is said to have a stable law if for all $n \geq 1$, there exist normalizing constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$X_1 + X_2 + \cdots + X_n \stackrel{d}{=} a_n X + b_n\,, \tag{1.2}$$

where $X_1, \ldots, X_n$ are independent copies of X and $\stackrel{d}{=}$ represents equality in distribution. Clearly, normal distributions satisfy (1.2). If X is non-Gaussian and yet satisfies (1.2), then the tail probability $P(|X| > x)$ behaves as $x^{-\alpha}$ for some $0 < \alpha < 2$, when x is large enough (see p. 16 in Samorodnitsky and Taqqu (1994)). A stable random variable X thus possesses the heavy-tail law in (1.1). To emphasize the dependence on the tail parameter α, X is often said to have an α-stable law.

Stable laws are closely related to the so-called generalized central limit theorem (Gnedenko and Kolmogorov (1954), Embrechts et al. (1997)). The generalized central limit theorem asserts that a class of stable laws coincides with that of all possible limit laws for properly normalized and centered sums of i.i.d. random variables. This indicates that, in heavy-tail analysis, stable laws play a fundamental role equivalent to that of normal distributions in light-tail models. Applications of stable distributions can be found in numerous publications. Early studies include those of Mandelbrot (1963) and Fama (1965), who applied stable distributions to financial data, and Paulson et al. (1975). A later study was conducted by, for instance, Nolan (2001).

Zolotarev (1986) is an encyclopedic book, which covers numerous topics such as an asymptotic expansion of stable densities and an inference problem of stable parameters. Janicki and Weron (1994) focused primarily on the simulation of stable distributions. Samorodnitsky and Taqqu (1994) is another encyclopedic book, covering a wide range of topics such as multivariate stable laws and the sample path properties of stable processes.

1.2 Long Range Dependence

In a stochastic process $(X(t),\ t \geq 0)$, long range dependence (sometimes called long memory) refers to non-negligible dependence between the current value $X(t)$ and initial value $X(0)$, which persists even as t increases. Long range dependence was empirically observed for the first time by Hurst in the 1950s (Hurst (1951, 1955)). Involved in a dam design project and studying the annual data of the water level of the Nile River, Hurst noticed what is now known as the Hurst phenomenon. Specifically, he discovered that the R/S statistics (= adjusted range / standard deviation) grows much faster than what is expected from independently and identically distributed observations. The Hurst phenomenon attracted much theoretical interest, culminating in a stochastic model with fractional Gaussian noise proposed by Mandelbrot and Van Ness (1968), which can theoretically explain the Hurst phenomenon.

Over the past several decades, long range dependence has been observed in diverse research areas such as DNA sequencing (Karmeshu and Krishnamachari (2004)), finance (Lobato and Velasco (2000), Lo (1997)), data networks (Beran et al. (1995), Karagiannis et al. (2004)), and climate (Fanchiotti et al. (2004), Varotsos and Kirk-Davidoff (2006)). For a historical background of long range dependence, we refer to Samorodnitsky (2006).

Long range dependence has historically been defined by the second order properties of stochastic processes. The most widely used definition is that, given a stochastic process $(X_n,\, n \geq 0)$ with finite variance, the process is a short memory process if the correlation function $\rho(X_n, X_0)$ is absolutely summable; otherwise, (X_n) is a long memory process. Because of the one-to-one correspondence between autocovariances and their spectral densities, long/short range dependence can also be defined in terms of behavior of the spectral density at the origin.

On the other hand, the classical definition of long/short range dependence becomes ambiguous when applied to heavy-tailed processes. In particular, using the second order property to distinguish between long and short range dependence is nonsensical if the process has infinite variance. To overcome this problem, several authors have proposed alternative covariance-like approaches such as covariation (Miller (1978)) and codifference (Kokoszka and Taqqu (1994)). However, the amount of information captured by these notions appears to be limited; see Samorodnitsky (2006) for more details.

As an alternative to those covariance-like functions, we shall adopt a recently developed, ergodic theoretical approach in categorizing long/short range dependence. The basic notions of ergodic theory on which our approach is based are summarized in, for example, Samorodnitsky (2004) and Roy (2008). We will explain those basics in more detail in Chapter 2.

1.3 Three Limit Theorems for Heavy-tailed Long Memory Infinitely Divisible Processes

A random vector $X \in \mathbb{R}^d$ is said to be infinitely divisible if for all $n \in \mathbb{N}$, there exist i.i.d. random vectors $Y_1, \dots, Y_n \in \mathbb{R}^d$ such that

$$X \stackrel{d}{=} Y_1 + \cdots + Y_n \,.$$

For an index set T, a stochastic process $(X(t),\, t \in T)$ is called an infinitely divisible process if for all $k \geq 1$ and $t_1, \dots, t_k \in T$, $\big(X(t_1) \dots X(t_k)\big)$ forms an infinitely divisible random vector. A class of infinitely divisible processes includes not only well-known Lévy processes, but also a wide variety of processes such as the Gaussian and the stable processes, moving averages driven by Lévy processes (e.g., Ornstein-Uhlenbeck processes), and harmonizable processes. Sato (1999) covers a range of topics on infinitely divisible distributions. More information on infinitely divisible processes and their integral representations is given in Rajput and Rosiński (1989).

Throughout this dissertation, we consider stationary heavy-tailed infinitely divisible processes of the form

$$X_n = \int_E f \circ T^n(x) dM(x) \,, \quad n = 1, 2, \dots \tag{1.3}$$

As shall be rigorously argued in Chapter 2, the length of memory in the process $\mathbf{X} = (X_1, X_2, \dots)$ is characterized by ergodic properties of the sequence of the operators (T^n). Our interest is in the process that is classified as a long memory process in such an ergodic theoretical sense. Our main goal is to establish three limit theorems for the process $\mathbf{X}$; the functional central limit theorem, the limit theorem on the sample autocovariance, and the functional limit theorem on the partial maxima. In the subsections below, we review how these limit theorems have evolved and discuss how our research relates to the previous work.

1.3.1 Functional Central Limit Theorems

Let $\mathbf{X} = (X_1, X_2, \dots)$ be a discrete time stationary stochastic process. A (functional) central limit theorem for such a process is a statement of the type

$$\left(\frac{1}{c_n} \sum_{k=1}^{\lceil nt \rceil} X_k - h_n t,\ 0 \le t \le 1 \right) \Rightarrow \Big(Y(t),\ 0 \le t \le 1 \Big) . \tag{1.4}$$

Here (c_n) is a positive sequence growing to infinity, (h_n) a real sequence, and $\Big(Y(t),\ 0 \le t \le 1\Big)$ is a non-degenerate (i.e. non-deterministic) process. Convergence in (1.4) is at least in finite dimensional distributions, but preferably it is weak convergence in the space $D[0,1]$ equipped with an appropriate topology. Not every stochastic process satisfies a central limit theorem, and for those that do, it is well known that both the rate of growth of the scaling constant c_n and the nature of the limiting process $\mathbf{Y} = \big(Y(t),\ 0 \le t \le 1\big)$ are determined both by the marginal tails of the stationary process $\mathbf{X}$ and its dependence structure. The limiting process (under very minor assumptions) is necessarily self-similar with stationary increments; this is known as the Lamperti theorem; see Lamperti (1962).

If, say, X_1 has a finite second moment, and $\mathbf{X}$ is an i.i.d. sequence then, clearly, one can choose $c_n = n^{1/2}$, and then $\mathbf{Y}$ is a Brownian motion. With equally light marginal tails, if the memory is sufficiently short, then one expects the situation to remain, basically, the same, and this turns out to be the case. When the variance is finite, the basic tool to measure dependence is, obviously, the correlations, which have to decay fast enough. It is well known, however, that a fast decay of correlations is alone not sufficient for this purpose, and, in general, certain strong mixing conditions have to be assumed. See for example Rosenblatt (1956) and, more recently, Merlevéde et al. (2006). If the memory is not sufficiently short, then both the rate of growth of c_n can be different from $n^{1/2}$, and the limiting process can be different from the Brownian motion. In fact, the limiting process may fail to be Gaussian at all; see e.g. Dobrushin and Major (1979) and Taqqu (1979).

If the marginal tails of the process are heavy, which, in this case, means that X_1 is in the domain of attraction of an α-stable law, $0 < \alpha < 2$, and $\mathbf{X}$ is an i.i.d. sequence then, clearly, one can choose c_n to be the inverse of the marginal tail (this makes c_n vary regularly with exponent $1/\alpha$), and then $\mathbf{Y}$ is an α-stable Lévy motion. Again, one expects the situation to remain similar if the memory is sufficiently short. Since correlations do not exist under heavy tails, statements of this type have been established for special models, often for moving average models; see e.g. Davis and Resnick (1985), Avram and Taqqu (1992) and Paulauskas and Surgailis (2008). Once again, as the memory gets longer, then both the rate of growth of c_n can be different from that obtained by inverting the marginal tail, and the limiting process will no longer have independent increments (i.e. be an α-stable Lévy motion). It is here, however, that the picture gets more interesting than in the case of light tails. First of all, in absence of correlations there is no canonical way of measuring how much longer the memory gets. Even more importantly, certain types of memory turn out to result in the limiting process $\mathbf{Y}$ being a self-similar α-stable process with stationary increments of a canonical form, the so-called Linear Fractional Stable motion; see e.g. Maejima (1983) for an example of such a situation, and Samorodnitsky and Taqqu (1994) for information on self-similar processes. However, when the memory gets even longer, Linear Fractional Stable motions disappear as well, and even more "unusual" limiting processes $\mathbf{Y}$ may appear. This phenomenon may qualify as change from short to long memory; see Samorodnitsky (2006).

In Chapter 3, we consider a functional central limit theorem for a class of heavy tailed stationary process exhibiting long memory in this sense. It is particularly interesting both because of the manner in which memory in the process is measured, and because the limiting process $\mathbf{Y}$ is a new class of stable self-similar processes with stationary increments. Interestingly, under certain parameter choices, $\mathbf{Y}$ happens to be an extension of a recently discovered local-time fractional stable motion (see Dombry and Guillotin-Plantard (2009)). We remark that Jung (2011) has proposed the so-called indicator fractional stable motions, which exhibit long memory in the same sense as the process by Dombry and Guillotin-Plantard (2009).

1.3.2 Limit Theorems of Sample Autocovariances and Sample Autocorrelations

For a discrete stationary process $(X_n,\, n \geq 0)$, the sample autocovariance function and the sample autocorrelation function are vital statistics in the analysis of dependence structure of the process. According to the Wold decomposition (see p. 187 in Brockwell and Davis (1991)), every stationary process with zero mean and finite variance can be represented by the sum of an infinite-order moving average and a perfectly predictable process. This fact justifies, to some extent, that every stationary process of finite second moment can be approximated by a moving average process (or equivalently, an ARMA(p, q) process of finite order). Thus, in a classical L^2-context, linear models are sufficient for data analysis; indeed, the sample autocorrelation function has traditionally been an important model-fitting and diagnostic tools (see, for example, Chapter 7 of Brockwell and Davis (1991)).

If stationary processes lack finite variance, they cannot generally be approximated by linear processes. Thus, it is natural to question whether classical methods based on sample autocorrelations are still plausible. For instance, a major feature of heavy tail models is that the sample autocorrelation converges to a *random* limit. If a random limit actually occurs, one needs to be more careful in applying traditional model-fitting and diagnostic tools such as the Akaike Information Criterion or Yule-Walker estimators. For more details, see Davis and Resnick (1996), Resnick and Van Den Berg (2000) and Resnick et al. (1999).

To determine the limit behavior of the sample autocovariances of infinite variance stationary processes, it is also important to see how rapidly the sample autocovariances grow. Many studies have revealed that if the tail of a marginal distribution is regularly varying with index $-\alpha$ for some $0 < \alpha < 2$, then a proper normalizing sequence (c_n) for the sample autocovariances may be written as $c_n = n^{1-2/\alpha}L(n)$, where $L(n)$ is a slowly varying function. Among the processes that possess such type of normalizing sequence are the linear process whose noise distribution has a balanced regularly varying tail (Davis and Resnick (1986)), the bilinear process (Davis and Resnick (1996), Resnick and Van Den Berg (2000)), certain ARCH processes (Davis and Mikosch (1998)) and α-stable moving average processes (Resnick et al. (1999)).

Resnick et al. (2000) reported an interesting phenomenon with respect to the growing rate of the sample autocovariance. They considered a process of the form (1.3), where $E = \mathbb{Z}^{\mathbb{N}}$, M is a symmetric α-stable random measure, and $T(x_0, x_1, \dots) = (x_1, x_2, \dots)$ is the left shift map defined on $\mathbb{Z}^{\mathbb{N}}$. Furthermore, M is assumed to have a control measure of the form $\mu(A) = \sum_{i\in\mathbb{Z}} \pi_i P_i(A)$, where $P_i(\cdot)$ is a probability law of an irreducible, null recurrent Markov chain with state space $\mathbb{Z}$, and (π_i) is its unique (up to multiplicative factors) σ-finite and invariant measure. By introducing an extra parameter $0 \le \beta \le 1$, they proved that a proper normalizing sequence in this situation is $c_n = n^{(1-\beta)(1-\alpha/2)}L(n)$. The parameter β accounts for the significantly longer memory of this process, relative to the other processes described in the previous paragraph; more details can be found in Samorodnitsky (2005).

An obvious drawback of the process studied by Resnick et al. (2000) is the highly specific form of the process and its control measure. In Chapter 4, we propose a more general framework inspired by the infinite ergodic theory, in which the asymptotics of the sample autocovariances can be more comprehensively assessed. In terms of the growth rate of the sample autocovariance and its weak limit, we will demonstrate that results similar to those of Resnick et al. (2000) are obtainable in the generalized framework.

1.3.3 Functional Limit Theorems for Partial Maxima Processes

Given a discrete stationary sequence $(X_n,\, n \geq 1)$, we will investigate the limit theorems on the partial maxima $\max_{1\leq k\leq n} |X_k|$, $n = 1, 2, \dots$. The most famous classical result of the limit behavior of the partial maxima is known as the Fisher Tippett theorem (see Fisher and Tippett (1928)). According to this theorem, if (X_n) are i.i.d. random variables, properly normalized partial maxima will converge to either a Fréchet, a Weibull, or a Gumbel distribution. In particular, if the distribution function of X_1 has a regularly varying tail with index $-\alpha$ for some $\alpha > 0$, the limiting distribution must be Fréchet. Since this dissertation aims to study heavy-tailed processes, we will focus only on the Fréchet limit case. The maxima dynamics of a stationary but non i.i.d. sequence are treated in detail in Leadbetter et al. (1983). These authors formulated several conditions, under which the extremes of the stationary sequence display the same qualitative behavior as that of an associated i.i.d. sequence. Leadbetter (1983) introduced the so-called extremal index, which can be regarded as a reciprocal of the mean cluster size. In the context of limit theorems of partial maxima for non i.i.d. random variables, extremes of moving averages have been discussed by many authors;

see, for example, Rootzén (1978), Davis and Resnick (1985), and Fasen (2005). Mikosch and Stărică (2000) investigated the extremes of GARCH processes.

A Fréchet process appears as a weak limit when the partial maxima process $(\max_{1 \le k \le \lfloor nt \rfloor} |X_k|,\, t \ge 0)$ is concerned. A stochastic process $(Y(t),\, t \in T)$ is called a Fréchet process if for all $n \ge 1$, $a_1, \dots, a_n > 0$ and $t_1, \dots, t_n \in T$, $\max_{1 \le j \le n} a_j Y(t_j)$ follows a Fréchet law. Recently, the structure of Fréchet processes, including their ergodic properties, has been gradually revealed; see Stoev (2008), Kabluchko et al. (2009), Wang and Stoev (2010). The spectral representations of Fréchet processes have also been extensively studied by Stoev and Taqqu (2005) and Kabluchko and Stoev (2012).

By adopting an ergodic theoretical approach, Samorodnitsky (2004) decomposed a stationary symmetric α-stable process (hereafter SαS process) into two independent processes; one with short range dependence and the other with long range dependence. He proved that if SαS processes exhibit long range dependence, their partial maxima grow strictly slower than $n^{1/\alpha}$. He also expressed SαS processes by a series representation, whose crucial components are arrival times $(\Gamma_j,\, j = 1, 2, \dots)$ of a unit rate Poisson process. Assuming that a single Poisson jump contributes to the maximum, Samorodnitsky concluded that the weak limit of the partial maxima has a Fréchet law.

Given the stationary process $\mathbf{X}$ in (1.3) and adopting the single Poisson jump assumption, the limit behavior of the partial maxima $\max_{1 \le k \le \lfloor nt \rfloor} |X_k|$, $n = 1, 2, \dots, t \ge 0$ is investigated in Chapter 5. As expected, the limiting process turns out to be some Fréchet process. Interestingly, however, it is parametrized not merely by heavy tailedness of the marginal distributions of the process $\mathbf{X}$, but also by the length of memory in $\mathbf{X}$.

1.4 Tail Measures

The heavy-tail assumption (1.1) can be extended to a multivariate form as follows. Let $\mathbf{X}$ be a d-dimensional random vector. $\mathbf{X}$ is said to have a regularly varying tail if there exists a function $H : (0, \infty) \mapsto (0, \infty)$ growing to infinity, and a nonzero Radon measure μ on $\overline{\mathbb{R}}^k \setminus \{\mathbf{0}\} = [-\infty, \infty]^k \setminus \{\mathbf{0}\}$ with $\mu(\overline{\mathbb{R}}^k \setminus \mathbb{R}^k) = 0$, such that

$$H(u)P\left(u^{-1}\mathbf{X} \in \ \cdot\ \right) \xrightarrow{v} \mu(\cdot) \tag{1.5}$$

vaguely in $\overline{\mathbb{R}}^k \setminus \{\mathbf{0}\}$. Various alternative definitions to (1.5) exist. Among the most popular is that there exists a random vector Θ on a unit sphere $\mathbb{S}^{k-1} = \{x \in \mathbb{R}^k : |x| = 1\}$, such that

$$\frac{P\big(|\mathbf{X}| > ux,\, \mathbf{X}/|\mathbf{X}| \in \cdot\big)}{P(|\mathbf{X}| > u)} \Rightarrow x^{-\alpha}P(\Theta \in \cdot)$$

weakly in $\mathbb{S}^{k-1}$. The probability measure $P \circ \Theta^{-1}$ is said to be *a spectral measure*. For other alternative definitions, we refer to Basrak et al. (2002a) and Resnick (2007).

Researchers are becoming increasingly interested in developing the statistics to measure dependence on extremes of stochastic processes. To this end, Ledford and Tawn (2003) introduced the so-called *upper tail dependence coefficient*; for a stationary sequence $(X_n,\, n \geq 0)$ of random variables,

$$\lambda(n) = \lim_{x\to\infty} P\big(X_n > x \,|\, X_0 > x\big)\,. \tag{1.6}$$

Under the multivariate regular variation condition (1.5), Davis and Mikosch (2009) considered *extremograms*; for a stationary sequence $(\mathbf{X}_n,\, n \geq 0)$ of $\mathbb{R}^d$-valued random vectors,

$$\gamma_{AB}(n) = \lim_{u\to\infty} H(u)^{-1}P\big(u^{-1}\mathbf{X}_0 \in A,\, u^{-1}\mathbf{X}_n \in B\big)\,, \tag{1.7}$$

$$\rho_{AB}(n) = \lim_{u\to\infty} P\big(u^{-1}\mathbf{X}_n \in B \,|\, u^{-1}\mathbf{X}_0 \in A\big)\,, \tag{1.8}$$

where $H : (0, \infty) \to (0, \infty)$ is a regularly varying function, and both A and $A \times B$ are Borel sets bounded away from zero. Resnick (2004) analyzed the alternative notion of *extreme dependence measure*. Fasen (2010) provides an elegant review of these types of notions.

However, all of these measures mainly describe dependence on extremes between two vectors only, $\mathbf{X}_0$ and $\mathbf{X}_n$. We are thus naturally motivated to develop the statistics for describing high-level dependence of the *whole* process $\mathbf{X}$. Let T be a (possibly infinite) parameter space and $\mathbf{X} = (X_t,\, t \in T)$ be a stochastic process or a random field. We further assume that $\mathbf{X}$ has regularly varying tails. That is, for all $k \geq 1$ and $t_1, \dots, t_k \in T$, the random vector $\big(X_{t_1}, \dots, X_{t_k}\big)$ has a regularly varying tail with limiting measure $\mu_{t_1 \dots t_k}$. In Chapter 6, we shall define a cylindrical measure ν on $\mathbb{R}^T$, termed *a tail measure*, such that for all $k \geq 1$ and $t_1, \dots, t_k \in T$,

$$\nu\Big\{x \in \mathbb{R}^T : (x_{t_1}, \dots, x_{t_k}) \in B\Big\} = \mu_{t_1 \dots t_k}(B)\,, \quad B \subseteq \mathbb{R}^k \setminus \{\mathbf{0}\}\,. \tag{1.9}$$

The measure ν can be seen to be an infinite-dimensional extension of a family of Radon measures $(\mu_{t_1 \dots t_k} :\, t_1, \dots, t_k \in T, k \geq 1)$.

Basrak and Segers (2009) proposed a related infinite-dimensional object, called *a tail process*, which contains information on high-level dependence for multivariate time series models. The notion of regular variation in $\mathbb{R}^d$ has been extended to probability laws on non-locally compact metric spaces (e.g. $\mathbb{C}[0,1]$, $\mathbb{D}[0,1]$); see Hult and Lindskog (2005, 2006). Such an extension, however, requires that usual vague convergence be replaced by the so-called $\widehat{w}$-convergence ($\widehat{w}$-convergence is detailed in Daley and Vere-Jones (2003)).

1.5 Outline of Dissertation

As mentioned earlier, we study the properties of stationary heavy-tailed, long memory, infinitely divisible processes of the form (1.3). A main characteristic of this dissertation is to adopt a recently developed, ergodic theoretical approach.

In Chapter 2, we overview basic notions on the ergodic theory. We also establish several novel results, namely *a generalized Darling-Kac theorem* and its variants, which will prove crucial in subsequent chapters.

Chapter 3 discusses the functional central limit theorems of the process in (1.3). We find that the limiting processes constitute a new class of symmetric stable self-similar processes with stationary increments. Under certain parameter choices, the limiting process is identical to that of Dombry and Guillotin-Plantard (2009). Chapter 4 treats limit theorems for sample autocovariances. Both the normalizing sequence and the weak limit determined by our ergodic framework are similar to those of Resnick et al. (2000). In Chapter 5, we prove the functional limit theorems on the partial maxima, under the single Poisson jump assumption. Throughout Chapters 3 to 5, the length of the memory in the process is parametrized by a specific assumption on the map T. Such parametrization enables us to understand more clearly the connection between the length of the memory in the process and the properties of the limiting processes.

Chapter 6 investigates a new notion called *a tail measure*. The first section provides a rigorous definition of a tail measure as a cylindrical measure ν given in (1.9). Subsequently, we study several properties of tail measures. We will then make use of the fact that, as an infinite-dimensional measure defined on *a big* space $\mathbb{R}^T$ (T is an arbitrary parameter space), tail measures have similar structure as (function level) Lévy measures of infinitely divisible processes. In fact, our argument is heavily inspired by an instructive lecture note on infinitely divisible processes by Rosiński (2007), which itself is partly based on Maruyama (1970). Section 6.2 covers several examples of tail measures for moving averages, independent processes, stochastic volatility processes and GARCH processes. Section 6.3 reveals the relation between tail measures and other related notions such as extremograms and tail processes. Finally, focusing on stationary infinitely divisible processes of the form (1.3), we will investigate the connection between the ergodic theoretical properties of tail measures and those of the probability laws of the processes.

CHAPTER 2

SOME ERGODIC THEORY

2.1 Preliminaries on Ergodic Theory

To rigorously characterize the memory properties of the process given in (1.3), we present some necessary elements of ergodic theory. The main reference for these notions is Aaronson (1997); see also Zweimüller (2009).

Let $(E, \mathcal{E}, \mu)$ be a σ-finite measure space. We will often use the notation $A = B$ mod μ for $A, B \in \mathcal{E}$ when $\mu(A \triangle B) = 0$.

Let $T : E \to E$ be a measurable map. T is called *nonsingular* if $\mu \circ T^{-1}$ and μ are equivalent. If $\mu \circ T^{-1}$ and μ coincide, T is said to be *a measure preserving map*. When the entire sequence $T, T^2, T^3, \dots$ of iterates of T is involved, we will sometimes refer to it as *a flow*. The map T is called *ergodic* if the only sets A in $\mathcal{E}$ for which $A = T^{-1}A$ mod μ are those for which $\mu(A) = 0$ or $\mu(A^c) = 0$. The map T is called *conservative* if

$$\sum_{n=1}^{\infty} \mathbf{1}_A \circ T^n = \infty \text{ a.e. on } A$$

for every $A \in \mathcal{E}$ with $\mu(A) > 0$. If T is ergodic, then the qualification "on A" above is not needed. For a nonsingular map T on a σ-finite measure space $(E, \mathcal{E}, \mu)$, E can be partitioned into two measurable subsets C and D, such that the map T is conservative on C and $D = E \setminus C$. We refer to C and D as *a conservative part* and *a dissipative part*. This decomposition is called *the Hopf decomposition*. A measurable set $W \in \mathcal{E}$ is *a wandering set* if $T^{-k}W$, $k \geq 0$, are disjoint. If T is nonsingular and invertible, D can be written as $D = \bigcup_{k \in \mathbb{Z}} T^{-k}W$ by some wandering set W. We refer the reader to Section 1.1 in Aaronson (1997) for more information on the Hopf decomposition.

The dual operator $\widehat{T}$ is an operator $L^1(\mu) \to L^1(\mu)$ defined by

$$\widehat{T}f = \frac{d(\nu_f \circ T^{-1})}{d\mu},$$

with ν_f a signed measure on $(E, \mathcal{E})$ given by $\nu_f(A) = \int_A f \, d\mu$, $A \in \mathcal{E}$. The dual operator satisfies the relation

$$\int_E \widehat{T} f \cdot g \, d\mu = \int_E f \cdot g \circ T \, d\mu \tag{2.1}$$

for $f \in L^1(\mu)$, $g \in L^\infty(\mu)$. For any nonnegative measurable function f on E a similar definition gives a nonnegative measurable function $\widehat{T} f$, and (2.1) holds for any two nonnegative measurable functions f and g.

An ergodic conservative measure preserving map T is called *pointwise dual ergodic* if there is a sequence of positive constants $a_n \nearrow \infty$ such that

$$\frac{1}{a_n} \sum_{k=1}^{n} \widehat{T}^k f \to \int_E f \, d\mu \ \text{ a.e.} \tag{2.2}$$

for every $f \in L^1(\mu)$. If the measure μ is infinite, pointwise dual ergodicity rules out invertibility of the map T; in fact no factor of T can be invertible, see p. 129 of Aaronson (1997).

We often require that the above convergence takes place uniformly on a set of finite measure. Let $A \in \mathcal{E}$ with $0 < \mu(A) < \infty$. A is said to be *a uniform set* for a conservative ergodic and measure preserving map T, if there exist a normalizing sequence $a_n \nearrow \infty$ and a nonnegative measurable function $f \in L^1(\mu)$ such that

$$\frac{1}{a_n} \sum_{k=1}^{n} \widehat{T}^k f \to \mu(f) \quad \text{uniformly, a.e. on } A\,. \tag{2.3}$$

If a measurable function f in (2.3) can be replaced by an indicator function $\mathbf{1}_A$, the set A is particularly called *a Darling-Kac set*. That is,

$$\frac{1}{a_n} \sum_{k=1}^{n} \widehat{T}^k \mathbf{1}_A \to \mu(A) \ \text{ uniformly, a.e. on } A. \tag{2.4}$$

From the similar argument as the proof of Proposition 3.7.5 in Aaronson (1997), one can see that T is pointwise dual ergodic if and only if T admits a uniform set. It is important to note that it is legitimate to use the same sequence (a_n) both in (2.2) and (2.3).

We often need to put a more strict assumption than (2.3). Let $A \in \mathcal{E}$ with $0 < \mu(A) < \infty$. A is said to be *a uniformly returning set* for a conservative ergodic and measure preserving map T, if there exist a normalizing sequence $b_n \nearrow \infty$ and a nonnegative measurable function $f \in L^1(\mu)$ such that

$$b_n \widehat{T}^n f \to \mu(f) \quad \text{uniformly, a.e. on } A\,. \tag{2.5}$$

Clearly any uniformly returning set is a uniform set. Further information on uniformly returning sets is given, for example, in Kesseböhmer and Slassi (2007).

Given a set $A \in \mathcal{E}$, the map $\varphi : E \to \mathbb{N} \cup \{\infty\}$ defined by $\varphi(x) = \inf\{n \geq 1 : T^n x \in A\}$, $x \in E$ is called *the first entrance time to A*. If T is conservative and ergodic (in addition to being measure preserving), and $\mu(A) > 0$, then $\varphi < \infty$ a.e. on E. It is natural to measure how often the set A is visited by the flow (T^n) by *the wandering rate* sequence

$$w_n = \mu\left(\bigcup_{k=0}^{n-1} T^{-k} A\right), \; n = 1, 2, \ldots .$$

There are several alternative expressions for the wandering rate sequence, the last two following from the fact that T is measure preserving.

$$w_n = \sum_{k=0}^{n-1} \mu(A_k) = \sum_{k=0}^{n-1} \mu\big(A \cap \{\varphi > k\}\big) = \sum_{k=1}^{\infty} \min(k, n) \mu\big(A \cap \{\varphi = k\}\big)\,. \tag{2.6}$$

Here $A_0 = A$ and $A_k = A^c \cap \{\varphi = k\}$ for $k \geq 1$. If μ is an infinite measure, T is conservative and ergodic, and $0 < \mu(A) < \infty$, then it follows from (2.6) that

$$w_n \sim \mu(\varphi < n) \quad \text{as } n \to \infty\,. \tag{2.7}$$

Let T be a conservative ergodic measure preserving map. If a set A is a uniform set, then there is a precise connection between the return sequence (w_n) and the normalizing sequence (a_n) in (2.3) (and, hence, also in (2.2)), assuming regular variation. Specifically, if either $(w_n) \in RV_{1-\beta}$ or $(a_n) \in RV_\beta$ for some $\beta \in [0, 1]$, then

$$a_n \sim \frac{1}{\Gamma(2-\beta)\Gamma(1+\beta)} \frac{n}{w_n} \quad \text{as } n \to \infty\,. \tag{2.8}$$

Proposition 3.8.7 in Aaronson (1997) gives one direction of this statement, but the argument is easily reversed. Analogously, a similar kind of connection between (w_n) and (b_n) in (2.5) is shown by Kesseböhmer and Slassi (2007). If either $(w_n) \in RV_{1-\beta}$ or $(b_n) \in RV_{1-\beta}$ for some $\beta \in (0,1]$, then

$$b_n \sim \Gamma(\beta)\Gamma(2-\beta)w_n \,. \tag{2.9}$$

2.2 Distributional Results in Ergodic Theory

In this section we prove three distributional ergodic theoretical results that will be used in the subsequent chapters. These results may be of interest on their own as well. We call our first result a generalized Darling-Kac theorem, because the first result of this type was proved in Darling and Kac (1957) as a distributional limit theorem for the occupation times of Markov processes and chains under a certain uniformity assumption on the transition law.

As a preparation for understanding the limiting law of our ergodic distributional results, we start with defining the Mittag-Leffler process. For $0 < \beta < 1$, let $\big(S_\beta(t), t \geq 0\big)$ be a β-stable subordinator, i.e. a Lévy process with increasing sample paths, satisfying $Ee^{-\theta S_\beta(t)} = \exp\{-t\theta^\beta\}$ for $\theta \geq 0$ and $t \geq 0$; see e.g. Chapter III of Bertoin (1996). Define its inverse process by

$$M_\beta(t) = S_\beta^{\leftarrow}(t) = \inf\big\{u \geq 0 : \; S_\beta(u) \geq t\big\}, \; t \geq 0 \,. \tag{2.10}$$

Recall that the marginal distributions of the process $\big(M_\beta(t), \, t \geq 0\big)$ are the Mittag-Leffler distributions, with the Laplace transform

$$E\exp\{\theta M_\beta(t)\} = \sum_{n=0}^{\infty} \frac{(\theta t^\beta)^n}{\Gamma(1+n\beta)}, \quad \theta \in \mathbb{R}; \tag{2.11}$$

see Proposition 1(a) in Bingham (1971). We will call this process *the Mittag-Leffler process*. This process has a continuous and non-decreasing version; we will always assume that we are working with such a version. It follows from (2.11) (or simply from the definition) that the Mittag-Leffler process is self-similar with exponent β. Further, all of its moments are finite. Recall, however, that this process has neither stationary nor independent increments; see e.g. Meerschaert and Scheffler (2004). A formal substitution of $\beta = 1$ into (2.11) indicates that $M_1(t)$ should be regarded as a straight line process, i.e. $M_1(t) = t$, $t \geq 0$. A straight line process can be viewed as the inverse of the degenerate 1-stable subordinator $S_1(t) = t$, $t \geq 0$. On the other hand, a formal substitution of $\beta = 0$ into (2.11) leads to a well-defined process $M_0(0) = 0$ and $M_0(t) = E$, the same standard exponential random variable for all $t > 0$. This process is no longer the inverse of a stable subordinator.

Under the same setup and assumptions as Darling and Kac (1957), Bingham (1971) extended their results to weak convergence in the space $D[0,\infty)$ endowed with the Skorohod J_1 topology, and the limiting process is the Mittag-Leffler process defined in (2.10). The result of Darling and Kac (1957) was put into ergodic-theoretic context by Aaronson (1981) who established the one-dimensional convergence for abstract conservative infinite measure preserving maps under the assumption of pointwise dual ergodicity, i.e. dispensing with a condition of uniformity. Furthermore, Aaronson proves convergence in a *strong distributional sense*, a stronger mode of convergence than weak convergence. The same strong distributional convergence was established later in Thaler and Zweimüller (2006), with the assumption of pointwise dual ergodicity replaced by a different type assumption involving the dynamics of the first entrance time to a certain reference set. The latter assumption was further weakened in Zweimüller (2007a). Our result, Theorem 2.2.1 below, extends Aaronson's result to the space $D[0,\infty)$, under the assumption of pointwise dual ergodicity.

We define strong distributional convergence. Let Y be a separable metric space, equipped with its Borel σ-field. Let $(\Omega_1, \mathcal{F}_1, m)$ be a measure space and $(\Omega_2, \mathcal{F}_2, P_2)$ a probability space. We say that a sequence of measurable maps $R_n : \Omega_1 \to Y$, $n =$

$1, 2, \ldots$ converges strongly in distribution to a measurable map $R : \Omega_2 \to Y$ if $P_1 \circ R_n^{-1} \Rightarrow P_2 \circ R^{-1}$ in Y for any probability measure $P_1 \ll m$ on $(\Omega_1, \mathcal{F}_1)$. That is,

$$\int_{\Omega_1} g(R_n)\, dP_1 \to \int_{\Omega_2} g(R)\, dP_2$$

for any such P_1 and a bounded continuous function g on Y. We will use the notation $R_n \stackrel{\mathcal{L}(m)}{\Rightarrow} R$ when strong distributional convergence takes place.

Theorem 2.2.1. ***(Generalized Darling-Kac Theorem)***

Let T be an ergodic conservative measure preserving map on an infinite σ-finite measure space $(E, \mathcal{E}, \mu)$. Assume that T is pointwise dual ergodic with a normalizing sequence (a_n) that is regularly varying with exponent $\beta \in (0, 1)$. Let $f \in L^1(\mu)$ be such that $\mu(f) = \int_E f d\mu \neq 0$, and denote $S_n(f) = \sum_{k=1}^n f \circ T^k$, $n = 1, 2, \ldots$. Then

$$\frac{1}{a_n} S_{\lceil n\cdot \rceil}(f) \stackrel{\mathcal{L}(\mu)}{\Rightarrow} \mu(f)\Gamma(1+\beta)M_\beta(\cdot) \quad \text{in } D[0,\infty)\,, \tag{2.12}$$

where M_β is the Mittag-Leffler process, and $D[0, \infty)$ is equipped with the J_1 topology.

Proof. It is shown in Corollary 3 of Zweimüller (2007b) that proving weak convergence in (2.12) for one fixed probability measure on $(E, \mathcal{E})$, that is absolutely continuous with respect to μ, already guarantees the full strong distributional convergence. We choose and fix an arbitrary set $A \in \mathcal{E}$ with $0 < \mu(A) < \infty$, and prove weak convergence in (2.12) with respect to $\mu_A(\cdot) = \mu(\cdot \cap A)/\mu(A)$.

It turns out that we only need to consider one particular function $f = \mathbf{1}_A$ and to establish the appropriate finite-dimensional convergence, i.e. to show that

$$\left(\frac{1}{a_n} S_{\lceil nt_i \rceil}(\mathbf{1}_A)\right)_{i=1}^k \Rightarrow \left(\mu(A)\Gamma(1+\beta)M_\beta(t_i)\right)_{i=1}^k \text{ in } \mathbb{R}^k \tag{2.13}$$

for all $k \geq 1$, $0 \leq t_1 < \cdots < t_k$, when the law of the random vector in the left hand side is computed with respect to μ_A.

Indeed, suppose that (2.13) holds. By Hopf's ergodic theorem (also sometimes called a ratio ergodic theorem; see Theorem 2.2.5 in Aaronson (1997)), the finite-dimensional convergence immediately extends to the corresponding finite-dimensional convergence with any function $f \in L^1(\mu)$ such that $\mu(f) \neq 0$. Next, write $f = f_+ - f_-$, the difference of the positive and negative parts. Since the process $\big(S_{\lceil nt \rceil}(f_+),\, t \geq 0\big)$ has, for each n, nondecreasing sample paths, Theorem 3 in Bingham (1971) tells us that the convergence of the finite-dimensional distributions, and the continuity in probability of the limiting Mittag-Leffler process already imply weak convergence, hence tightness, of this sequence of processes. Similarly, the sequence of the processes $\big(S_{\lceil nt \rceil}(f_-),\, t \geq 0\big)$, $n = 1, 2, \ldots$ is tight as well. Since both converge to a continuous limit, their sum, $\big(S_{\lceil nt \rceil}(f),\, t \geq 0\big)$, $n = 1, 2, \ldots$, is tight as well, because in this case the uniform modulus of continuity can be used instead of the J_1 modulus of continuity; see e.g. Billingsley (1999).

This will give us the required weak convergence and, hence, finish the proof of the theorem.

It remains to show (2.13). We will use a strategy similar to the one used in Bingham (1971). We start with defining a continuous version of the process $\big(S_{\lceil nt \rceil}(\mathbf{1}_A),\, t \geq 0\big)$ given by the linear interpolation

$$\tilde{S}_n(t) = \big((i+1) - nt\big) S_i(\mathbf{1}_A) + (nt - i) S_{i+1}(\mathbf{1}_A) \ \text{ if } \ \frac{i}{n} \leq t \leq \frac{i+1}{n},\ i = 0, 1, 2, \ldots . \tag{2.14}$$

With the implicit argument $x \in E$ viewed as random (with the law μ_A), each $\tilde{S}_n$ defines a random Radon measure on $[0, \infty)$. Therefore, for any $k \geq 1$ the k-tuple product $\tilde{S}_n^k = \tilde{S}_n \times \ldots \times \tilde{S}_n$ is a random Radon measure on $[0, \infty)^k$. By Fubini's theorem,

$$\tilde{m}_n^{(k)}(B) = \int_A \tilde{S}_n^k(B)(x)\, \mu_A(dx),\ B \subseteq [0, \infty)^k, \ \text{Borel},$$

is a Radon measure on $[0, \infty)^k$. We define, similarly, S_n, S_n^k and $m_n^{(k)}$, starting with $S_n(t) = S_{\lceil nt \rceil}(\mathbf{1}_A),\, t \geq 0$. Finally, we perform the same operation on the limiting process and define $M_{\beta,A}$ by $\mu(A)\Gamma(1+\beta)M_\beta$, and then construct $M_{\beta,A}^k$ and $m_{\beta,A}^{(k)} = EM_{\beta,A}^k$.

Note that $\tilde{m}_n^{(k)}$ is absolutely continuous with respect to the k-dimensional Lebesgue measure, and

$$\frac{d^k \tilde{m}_n^{(k)}}{dt_1 \dots dt_k} = n^k \int_A \prod_{j=1}^k \mathbf{1}_A \circ T^{i_j}(x)\, \mu_A(dx) \text{ on } \frac{i_j}{n} \le t_j < \frac{i_j+1}{n},\ i_j = 0, 1, \dots,\ j = 1, \dots, k\,.$$

We will prove that for all $k \ge 1$, $\theta_1, \dots, \theta_k \ge 0$,

$$\frac{1}{a_n^k} \int_0^\infty \!\!\dots \int_0^\infty e^{-\sum_{j=1}^k \theta_j t_j}\, \tilde{m}_n^{(k)}(dt_1, \dots dt_k) \to \int_0^\infty \!\!\dots \int_0^\infty e^{-\sum_{j=1}^k \theta_j t_j}\, m_{\beta,A}^{(k)}(dt_1, \dots dt_k) \tag{2.15}$$

as $n \to \infty$. We claim that this will suffice for (2.13).

Indeed, suppose that (2.15) holds. Convergence of the joint Laplace transforms implies that

$$a_n^{-k} \tilde{m}_n^{(k)} \xrightarrow{v} m_{\beta,A}^{(k)}$$

(vaguely) in $[0, \infty)^k$. Since the rectangles are, clearly, compact continuity sets with respect to the limiting measure $m_{\beta,A}^{(k)}$, we conclude that for every $k = 1, 2, \dots$ and $t_j \ge 0$, $j = 1, \dots, k$, we have

$$\int_A \prod_{j=1}^k a_n^{-1} \tilde{S}_n(t_j)(x)\, \mu_A(dx) = a_n^{-k} \tilde{m}_n^{(k)} \Big(\prod_{j=1}^k [0, t_j] \Big)$$

$$\to m_{\beta,A}^{(k)} \Big(\prod_{j=1}^k [0, t_j] \Big) = E\Big[\prod_{j=1}^k \mu(A) \Gamma(1+\beta) M_\beta(t_j) \Big]$$

as $n \to \infty$. Since for every fixed $\varepsilon > 0$ and $n > 1/\varepsilon$,

$$\tilde{S}_n(t) \le S_n(t) \le \tilde{S}_n(t + \varepsilon)$$

for each $t \ge 0$, we conclude by monotonicity and continuity of the Mittag-Leffler process that

$$\int_A \prod_{j=1}^k a_n^{-1} S_n(t_j)\, \mu_A(dx) \to E\Big[\prod_{j=1}^k \mu(A) \Gamma(1+\beta) M_\beta(t_j) \Big]\,. \tag{2.16}$$

We claim that (2.16) implies (2.13). By taking linear combinations with nonnegative weights, we see that it is enough to show that the distribution of such a linear combination,

$$\sum_{j=1}^k \theta_j M_\beta(t_j),\ \theta_j > 0,\ j = 1, \dots, k\,,$$

is determined by its moments, and by the Carleman sufficient condition it is enough to check that

$$\sum_{m=1}^{\infty}\left(\frac{1}{E\big(\sum_{j=1}^{k}\theta_j M_\beta(t_j)\big)^m}\right)^{1/(2m)} = \infty\,.$$

A simple monotonicity and scaling argument shows that it is sufficient to verify only that

$$\sum_{m=1}^{\infty}\left(\frac{1}{E\big(M_\beta(1)\big)^m}\right)^{1/(2m)} = \infty\,. \tag{2.17}$$

However, the moments of $M_\beta(1)$ can be read off (2.11), and Stirling's formula together with elementary algebra imply (2.17). Hence (2.13) follows.

It follows that we need to prove (2.15). Taking into account the form of the density of $\tilde{m}_n^{(k)}$ with respect to the k-dimensional Lebesgue measure, we can write the left hand side of (2.15) as

$$\sum_{\pi} F_{n,A}\big(\theta_{\pi(1)}\ldots\theta_{\pi(k)}\big)\,,$$

where

$$F_{n,A}(\theta_1\ldots\theta_k) = \left(\frac{n}{a_n}\right)^k \int\cdots\int_{0<t_1<\cdots<t_k} e^{-\sum_{j=1}^{k}\theta_j t_j}\mu_A\left(\bigcap_{j=1}^{k} T^{-\lceil nt_j\rceil}A\right) dt_1\ldots dt_k\,,$$

and π runs through the permutations of the sets $\{1,\ldots,k\}$. To establish (2.15), it is enough to verify that

$$F_{n,A}(\theta_1\ldots\theta_k) \to \big(\mu(A)\Gamma(1+\beta)\big)^k\big((\theta_1+\cdots+\theta_k)(\theta_2+\cdots+\theta_k)\ldots\theta_k\big)^{-\beta} \tag{2.18}$$

as $n\to\infty$, because Lemma 3 in Bingham (1971) shows that summing up the expression in the right hand side of (2.18) over all possible permutations $(\theta_{\pi(1)}\ldots\theta_{\pi(k)})$ produces the expression in the right hand side of (2.15).

Given $0<\varepsilon<1$, we use repeatedly pointwise dual ergodicity and Egorov's theorem to construct a nested sequence of measurable subsets of E, with $A_0=A$, and for $i=0,1,\ldots$, $A_{i+1}\subseteq A_i$, and $\mu(A_{i+1})\geq(1-\varepsilon)\mu(A_i)$, while

$$\frac{1}{a_n}\sum_{k=1}^{n}\widehat{T}^k\mathbf{1}_{A_i}\to\mu(A_i)\ \text{ uniformly on } A_{i+1}. \tag{2.19}$$

It is elementary to see that with $v_1 = \theta_1 + \theta_2 + \cdots + \theta_k$, $v_2 = \theta_2 + \cdots + \theta_k, \ldots, v_k = \theta_k$,

$$F_{n,A}(\theta_1 \ldots \theta_k) \sim \frac{1}{a_n^k} \sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} e^{-n^{-1}\sum_{j=1}^k v_j m_j} \mu_A \left(\bigcap_{j=1}^{k} T^{-(m_1+\ldots+m_j)} A \right) \tag{2.20}$$

$$= \frac{1}{a_n^k} \int_A \left[\left(\sum_{m_1=0}^{\infty} \widehat{T}^{m_1} \mathbf{1}_A \, e^{-v_1 m_1/n} \right) \prod_{j=2}^{k} \left(\sum_{m_j=0}^{\infty} \mathbf{1}_A \circ T^{m_2+\ldots+m_j} \, e^{-v_j m_j/n} \right) \right] d\mu_A$$

$$\geq \frac{1}{a_n^k} \int_{A_1} \left(\cdots \right),$$

where the equality is due to the duality relation (2.1). Note that by (2.19) with $i = 0$,

$$\sum_{m_1=0}^{\infty} \widehat{T}^{m_1} \mathbf{1}_A \, e^{-v_1 m_1/n} = \left(1 - e^{-v_1/n}\right) \sum_{i=0}^{\infty} \left(\sum_{m_1=0}^{i} \widehat{T}^{m_1} \mathbf{1}_{A_0} \right) e^{-v_1 i/n} \tag{2.21}$$

$$\sim \frac{\mu(A_0) v_1}{n} \sum_{i=0}^{\infty} a_i e^{-v_1 i/n}$$

uniformly on A_1 as $n \to \infty$. Therefore,

$$F_{n,A}(\theta_1 \ldots \theta_k) \geq \left(1 - o(1)\right) \frac{1}{a_n^k} \frac{\mu(A_0) v_1}{n} \sum_{i=0}^{\infty} a_i e^{-v_1 i/n}$$

$$\times \int_{A_1} \prod_{j=2}^{k} \left(\sum_{m_j=0}^{\infty} \mathbf{1}_A \circ T^{m_2+\ldots+m_j} \, e^{-v_j m_j/n} \right) d\mu_A$$

$$= \left(1 - o(1)\right) \frac{1}{a_n^k} \frac{\mu(A_0) v_1}{n} \sum_{i=0}^{\infty} a_i e^{-v_1 i/n}$$

$$\times \int_A \left[\left(\sum_{m_2=0}^{\infty} \widehat{T}^{m_2} \mathbf{1}_{A_1} \, e^{-v_2 m_2/n} \right) \prod_{j=3}^{k} \left(\sum_{m_j=0}^{\infty} \mathbf{1}_A \circ T^{m_3+\ldots+m_j} \, e^{-v_j m_j/n} \right) \right] d\mu_A$$

$$\geq \left(1 - o(1)\right) \frac{1}{a_n^k} \frac{\mu(A_0) v_1}{n} \sum_{i=0}^{\infty} a_i e^{-v_1 i/n} \int_{A_2} \left(\cdots \right).$$

Using now repeatedly (2.19) with larger and larger i, together with the same argument as in (2.21), we conclude that

$$F_{n,A}(\theta_1 \ldots \theta_k) \geq \left(1 - o(1)\right) \frac{1}{a_n^k} \frac{\mu(A_0)\mu(A_1) v_1 v_2}{n^2} \sum_{i_1=0}^{\infty} a_{i_1} e^{-v_1 i_1/n} \sum_{i_2=0}^{\infty} a_{i_2} e^{-v_2 i_2/n}$$

$$\times \int_{A_2} \prod_{j=3}^{k} \left(\sum_{m_j=0}^{\infty} \mathbf{1}_A \circ T^{m_3+\ldots+m_j} \, e^{-v_j m_j/n} \right) d\mu_A$$

$$\geq \cdots \geq \big(1-o(1)\big)\frac{1}{a_n^k}\frac{\prod_{j=0}^{k-1}\mu(A_j)v_{j+1}}{n^k}\prod_{j=1}^{k}\left(\sum_{i=0}^{\infty}a_i e^{-v_j i/n}\right)\frac{\mu(A_k)}{\mu(A)}$$

$$\geq \big(1-o(1)\big)(1-\varepsilon)^{k(k+1)/2}\left(\frac{\mu(A)}{na_n}\right)^k (v_1\dots v_k)\prod_{j=1}^{k}\left(\sum_{i=0}^{\infty}a_i e^{-v_j i/n}\right).$$

Extending the sequence (a_n) into a piece-wise constant regular varying function of real variable $\big(a(x),\, x>0\big)$ and using Karamata's Tauberian Theorem (see e.g. Corollary 1.7.3 in Bingham et al. (1987)), we conclude that for every $j=1,\dots,k$,

$$\sum_{i=0}^{\infty}a_i e^{-v_j i/n}\sim \Gamma(1+\beta)\frac{n}{v_j}a(n/v_j),\quad n\to\infty\,.$$

It follows that

$$F_{n,A}(\theta_1\dots\theta_k)\geq \big(1-o(1)\big)(1-\varepsilon)^{k(k+1)/2}\Big(\mu(A)\Gamma(1+\beta)\Big)^k\prod_{j=1}^{k}\frac{a(n/v_j)}{a_n}$$

$$\to (1-\varepsilon)^{k(k+1)/2}\Big(\mu(A)\Gamma(1+\beta)\Big)^k\prod_{j=1}^{k}v_j^{-\beta}$$

by the regular variation. Since this is true for every $0<\varepsilon<1$, we have obtained the lower bound

$$\liminf_{n\to\infty}F_{n,A}(\theta_1\dots\theta_k)\geq \big(\mu(A)\Gamma(1+\beta)\big)^k\big((\theta_1+\cdots+\theta_k)(\theta_2+\cdots+\theta_k)\dots\theta_k\big)^{-\beta}. \quad (2.22)$$

The lower bound (2.22) is valid for any measurable set A with $0<\mu(A)<\infty$. We will now show that for any $k\geq 1$ and $0<\theta<1$ there is a measurable set $A_{k,\theta}\subseteq A$ such that

$$\mu\big(A_{k,\theta}\big)\geq (1-\theta)\mu(A)\,, \quad (2.23)$$

and such that

$$\limsup_{n\to\infty}F_{n,A_{k,\theta}}(\theta_1\dots\theta_k)\leq \big(\mu(A_{k,\theta})\Gamma(1+\beta)\big)^k\big((\theta_1+\cdots+\theta_k)(\theta_2+\cdots+\theta_k)\dots\theta_k\big)^{-\beta}. \quad (2.24)$$

We know that (2.22) and (2.24) together imply (2.18), hence that (2.13) holds for the set $A_{k,\theta}$. We claim that this implies that (2.13) for every measurable A with $0<\mu(A)<\infty$.

Indeed, suppose that, to the contrary, (2.13) fails for some measurable A with $0<\mu(A)<\infty$, some $k\geq 1$ and some $0<t_1<\ldots<t_k$. By the one-dimensional

result of Aaronson (1981), the k components in the left hand side of (2.13), individually, converge weakly. Therefore, the sequence of the laws of the k-dimensional vectors in the left hand side of (2.13) is tight, and so there is a sequence of integers $n_l \uparrow \infty$ and a random vector $(Y_1, \dots, Y_k)$ with

$$(Y_1, \dots, Y_k) \overset{d}{\neq} \mu(A)\Gamma(1+\beta)\big(M_\beta(t_1)\dots M_\beta(t_k)\big)\,, \tag{2.25}$$

such that

$$\frac{1}{a_{n_l}}\big(S_{\lceil n_l t_1\rceil}(\mathbf{1}_A), \dots, S_{\lceil n_l t_k\rceil}(\mathbf{1}_A)\big) \Rightarrow (Y_1, \dots, Y_k)\,, \tag{2.26}$$

when the law of the random vector in the left hand side is computed with respect to μ_A. It follows from (2.25) that there is a Borel set $B \subset \mathbb{R}^k$ such that, for each $b > 0$, bB is a continuity set for both $(Y_1, \dots, Y_k)$ and $\mu(A)\Gamma(1+\beta)\big(M_\beta(t_1)\dots M_\beta(t_k)\big)$ and (abusing the notation a bit by using the same letter P),

$$P\Big(\mu(A)\Gamma(1+\beta)\big(M_\beta(t_1)\dots M_\beta(t_k)\big) \in B\Big) > (1+\rho)P\Big((Y_1, \dots, Y_k) \in B\Big) \tag{2.27}$$

for some $\rho > 0$. In fact, since the law of a Mittag-Leffler random variable is atomless, such a B can be taken to be either a "SW corner" of the type $B = \prod_{j=1}^k(-\infty, x_j]$ for some $(x_1, \dots, x_k) \in \mathbb{R}^k$, or its complement.

Choose now $0 < \theta < 1$ so small that

$$(1-\theta)(1+\rho) > 1\,, \tag{2.28}$$

and consider the set $A_{k,\theta}$. It follows from (2.26) and Hopf's ergodic theorem that

$$\frac{1}{a_{n_l}}\big(S_{\lceil n_l t_1\rceil}(\mathbf{1}_{A_{k,\theta}}), \dots, S_{\lceil n_l t_k\rceil}(\mathbf{1}_{A_{k,\theta}})\big) \Rightarrow \frac{\mu\big(A_{k,\theta}\big)}{\mu(A)}(Y_1, \dots, Y_k)\,,$$

when the law of the random vector in the left hand side is still computed with respect to μ_A. However, since (2.13) holds for the set $A_{k,\theta}$, we see that

$$P\Big((Y_1, \dots, Y_k) \in B\Big) = \lim_{l\to\infty} \mu_A\Big(\frac{1}{a_{n_l}}\big(S_{\lceil n_l t_1\rceil}(\mathbf{1}_{A_{k,\theta}}), \dots, S_{\lceil n_l t_k\rceil}(\mathbf{1}_{A_{k,\theta}})\big) \in \frac{\mu\big(A_{k,\theta}\big)}{\mu(A)}B\Big)$$

$$= \frac{\mu\big(A_{k,\theta}\big)}{\mu(A)} \lim_{l\to\infty} \mu_{A_{k,\theta}}\Big(\frac{1}{a_{n_l}}\big(S_{\lceil n_l t_1\rceil}(\mathbf{1}_{A_{k,\theta}}), \dots, S_{\lceil n_l t_k\rceil}(\mathbf{1}_{A_{k,\theta}})\big) \in \frac{\mu\big(A_{k,\theta}\big)}{\mu(A)}B\Big)$$

$$\geq (1-\theta)P\Big(\mu(A)\Gamma(1+\beta)\big(M_\beta(t_1)\dots M_\beta(t_k)\big) \in B\Big)$$

$$> P\big((Y_1,\dots,Y_k) \in B\big)\,,$$

where the last inequality follows from (2.27) and (2.28). This contradiction shows that, once we prove (2.24), this will establish (2.13) for every measurable A with $0 < \mu(A) < \infty$.

We call a nested sequence $(A_0, A_1, \dots)$ of sets in (2.19) an ε-sequence starting at A_0. Its finite subsequence $(A_0, A_1, \dots, A_k)$ will be called an ε-sequence of length $k+1$ starting at A_0 and ending at A_k. Let A be a measurable set with $0 < \mu(A) < \infty$. Fix $0 < \theta < 1$. Let $0 < r < 1$ be a small number, to be specified in the sequel. We construct a nested sequence of sets as follows.

Let $B_0 = A$. Construct an r-sequence of length $k+1$ starting at B_0, and ending at some set $B_1 \subseteq B_0$. Next, construct an r^2-sequence of length $k+1$ starting at B_1, and ending at some set $B_2 \subseteq B_1$. Proceeding this way we obtain a nested sequence of measurable sets $A = B_0 \supseteq B_1 \supseteq B_2 \supseteq \dots$, such that

$$\mu(B_n) \geq \prod_{i=1}^{n}(1-r^i)^k\,\mu(A),\ n = 1, 2, \dots\,.$$

The sets (B_n) decrease to some set $A_{k,\theta}$ with

$$\mu(A_{k,\theta}) \geq \prod_{i=1}^{\infty}(1-r^i)^k\,\mu(A)\,.$$

Notice that, by choosing $0 < r < 1$ small enough, we can ensure that (2.23) holds. Note, further, that by construction, for every $d = 1, 2, \dots$,

$$\mu(A_{k,\theta}) \geq f_d\,\mu(B_d),\ \text{ with }\ f_d = \prod_{i=d+1}^{\infty}(1-r^i)^k\,.$$

Clearly, $f_d \uparrow 1$ as $d \to \infty$. Starting with the first line in (2.20), we see that

$$F_{n,A_{k,\theta}}(\theta_1 \dots \theta_k)$$

$$\leq \big(1+o(1)\big)\frac{1}{a_n^k}\sum_{m_1=0}^{\infty}\dots\sum_{m_k=0}^{\infty} e^{-n^{-1}\sum_{j=1}^k v_j m_j}\mu_{B_d}\left(\bigcap_{j=1}^{k} T^{-(m_1+\ldots+m_j)}B_d\right)\frac{\mu(B_d)}{\mu\big(A_{k,\theta}\big)}$$

$$\leq \big(1+o(1)\big)\frac{1}{f_d}\frac{1}{a_n^k}$$

$$\times \int_{B_d}\left[\left(\sum_{m_1=0}^{\infty}\widehat{T}^{m_1}\mathbf{1}_{B_{d-1}}\, e^{-v_1 m_1/n}\right)\prod_{j=2}^{k}\left(\sum_{m_j=0}^{\infty}\mathbf{1}_{B_d}\circ T^{m_2+\ldots+m_j}\, e^{-v_j m_j/n}\right)\right]\, d\mu_{B_d}\,.$$

Using repeatedly uniform convergence as in (2.21) above, we conclude, as in the case of the corresponding lower bound calculation, that

$$F_{n,A_{k,\theta}}(\theta_1\ldots\theta_k)\leq \big(1+o(1)\big)\frac{1}{f_d}\frac{1}{a_n^k}\frac{\mu\big(B_{d-1}\big)v_1}{n}\sum_{i=0}^{\infty}a_i e^{-v_1 i/n}$$

$$\times \int_{B_d}\left[\left(\sum_{m_2=0}^{\infty}\widehat{T}^{m_2}\mathbf{1}_{B_{d-1}}\, e^{-v_2 m_2/n}\right)\prod_{j=3}^{k}\left(\sum_{m_j=0}^{\infty}\mathbf{1}_{B_d}\circ T^{m_3+\ldots+m_j}\, e^{-v_j m_j/n}\right)\right]\, d\mu_{B_d}$$

$$\leq\cdots\leq \big(1+o(1)\big)\frac{1}{f_d}\Big(\frac{\mu(B_{d-1})}{na_n}\Big)^k(v_1\ldots v_k)\prod_{j=1}^{k}\Big(\sum_{i=0}^{\infty}a_i e^{-v_j i/n}\Big)$$

$$\leq \big(1+o(1)\big)\frac{1}{f_d f_{d-1}^k}\Big(\frac{\mu(A_{k,\theta})}{na_n}\Big)^k(v_1\ldots v_k)\prod_{j=1}^{k}\Big(\sum_{i=0}^{\infty}a_i e^{-v_j i/n}\Big)\,.$$

As in the case of the lower bound, Karamata's Tauberian theorem shows that

$$F_{n,A_{k,\theta}}(\theta_1\ldots\theta_k)\leq \big(1+o(1)\big)\frac{1}{f_d f_{d-1}^k}\Big(\mu(A_{k,\theta})\Gamma(1+\beta)\Big)^k\prod_{j=1}^{k}\frac{a(n/v_j)}{a_n}$$

$$\to \frac{1}{f_d f_{d-1}^k}\Big(\mu(A_{k,\theta})\Gamma(1+\beta)\Big)^k\prod_{j=1}^{k}v_j^{-\beta}$$

as $n\to\infty$. Since this is true for every $d\geq 1$, we can let now $d\to\infty$ to obtain (2.23), and the proof of the theorem is complete. $\square$

Remark 2.2.2. It follows immediately from Theorem 2.2.1 and continuity of the limiting Mittag-Leffler process that for the continuous process $(\tilde{S}_n)$ defined in (2.14), strong distributional convergence as in (2.12) also holds, either in $D[0,\infty)$ or in $C[0,\infty)$.

We use the strong distributional convergence obtained in Theorem 2.2.1 in the following propositions. Proposition 2.2.3 below will apply in the proof of the main theorem in Chapter 3.

Proposition 2.2.3. *Under the assumptions of Theorem 2.2.1, let $A \in \mathcal{E}$, $0 < \mu(A) < \infty$, be a uniform set for T. Let $A_0 = A$, $A_k = A^c \cap \{\varphi = k\}$, $k \geq 1$, where φ is the first entrance time of A. Suppose that*

$$\frac{\widehat{T}^n \mathbf{1}_{A_n}}{\mu(A_n)} \to K \quad \textit{uniformly, a.e. on } A\,, \tag{2.29}$$

where $K : E \to \mathbb{R}_+$ is a measurable function, and that the function f is supported by A. Define a probability measure on E by $\mu_n(\cdot) = \mu(\cdot \cap \{\varphi \leq n\})/\mu(\{\varphi \leq n\})$. Let $0 \leq t_1 < \cdots < t_H$, $H \geq 1$, and fix $L \in \mathbb{N}$ with $t_H \leq L$. Then under μ_{nL},

$$\left(\frac{S_{\lceil nt_h \rceil}(f)}{a_n},\, h = 1, \ldots, H\right) \Rightarrow \big(\mu(f)\Gamma(1+\beta)M_\beta((t_h - T_\infty^{(L)})_+),\, h = 1, \ldots, H\big) \ \ \textit{in } \mathbb{R}^H\,,$$

where $T_\infty^{(L)}$ is a random variable independent of the Mittag-Leffler process M_β, with $P\big(T_\infty^{(L)} \leq x\big) = (x/L)^{1-\beta},\ 0 \leq x \leq L$.

Remark 2.2.4. The condition (2.29) is an extension of the property shared by certain operators T, the so-called Markov shifts (see Chapter 4 in Aaronson (1997)), to a more general class of operators.

Proof. Since T preserves measure μ, for the duration of the proof we may and will modify the definition of S_n to $S_n(f) = \sum_{k=0}^{n-1} f \circ T^k$, $n = 1, 2, \ldots$. Fix $\theta_1, \ldots, \theta_H \in \mathbb{R}$ and let $\lambda \in \mathbb{R}$. Since f is supported by A, we have, as $n \to \infty$,

$$\begin{aligned}
\mu_{nL}\left(\frac{1}{a_n}\sum_{h=1}^H \theta_h S_{\lceil nt_h \rceil}(f) > \lambda\right) &\sim \mu_{nL}\left(A^c \cap \left\{\frac{1}{a_n}\sum_{h=1}^H \theta_h S_{\lceil nt_h \rceil}(f) > \lambda\right\}\right) \\
&= \mu(\varphi \leq nL)^{-1} \sum_{m=1}^{nL} \mu\left(A_m \cap \left\{\frac{1}{a_n}\sum_{h=1}^H \theta_h S_{\lceil nt_h \rceil}(f) > \lambda\right\}\right) \\
&\sim \mu(\varphi \leq nL)^{-1} \sum_{m=1}^{nL} \mu\left(A_m \cap T^{-m}\left\{\frac{1}{a_n}\sum_{h=1}^H \theta_h S_{(\lceil nt_h \rceil - m)_+}(f) > \lambda\right\}\right) \\
&= \int_A \frac{1}{\mu(\varphi \leq nL)} \sum_{m=1}^{nL} \widehat{T}^m \mathbf{1}_{A_m} \cdot \mathbf{1}_{\{\sum_{h=1}^H \theta_h S_{(\lceil nt_h \rceil - m)_+}(f) > \lambda a_n\}} d\mu\,.
\end{aligned}$$

Note that the measure on E defined by $\eta(\cdot) = \int_\cdot K d\mu$ with K in (2.29) is necessarily a probability measure. We conclude by (2.29) that

$$\mu_{nL}\left(\frac{1}{a_n}\sum_{h=1}^H \theta_h S_{\lceil nt_h \rceil}(f) > \lambda\right) \sim \sum_{m=1}^{nL} \eta\left(\frac{1}{a_n}\sum_{h=1}^H \theta_h S_{(\lceil nt_h \rceil - m)_+}(f) > \lambda\right) p_n(m)\,, \tag{2.30}$$

where $p_n(j) = \mu(A_j)/\sum_{m=1}^{nL} \mu(A_m)$, $j = 1, \ldots, nL$, is a probability mass function. Let $T_n^{(L)}$ be a discrete random variable with this probability mass function, independent of $S_{\lceil n\cdot \rceil}(f)$, which is, in turn, governed by the probability measure η. If we declare that $T_n^{(L)}$ is defined on some probability space $(\Omega_n, \mathcal{F}_n, P_n)$, then the right hand side of (2.30) becomes

$$(\eta \times P_n)\left(\frac{1}{a_n}\sum_{h=1}^{H}\theta_h S_{(\lceil nt_h\rceil - T_n^{(L)})_+}(f) > \lambda\right).$$

Since η is a probability measure absolutely continuous with respect to μ, it follows from the strong distributional convergence in Theorem 2.2.1 that

$$\frac{1}{a_n} S_{\lceil n\cdot \rceil}(f) \Rightarrow \mu(f)\Gamma(1+\beta)M_\beta(\cdot) \quad \text{in } D[0,L], \tag{2.31}$$

when the law in the left hand side is computed with respect to η. On the other hand, by the regular variation of the wandering rate sequence and (2.7), for $x \in [0, L]$,

$$P_n\left(\frac{T_n^{(L)}}{n} \le x\right) = \sum_{m=1}^{\lceil nx\rceil} p_n(m) \sim \frac{w_{\lceil nx\rceil}}{w_{nL}} \sim \left(\frac{x}{L}\right)^{1-\beta}, \tag{2.32}$$

which is precisely the law of $T_\infty^{(L)}$. We can put together (2.31), (2.32), and independence between S_n and $T_n^{(L)}$ to obtain

$$\mu_{nL}\left(\frac{1}{a_n}\sum_{h=1}^{H}\theta_h S_{\lceil nt_h\rceil}(f) > \lambda\right) \to P\left(\mu(f)\Gamma(1+\beta)\sum_{h=1}^{H}\theta_h M_\beta((t_h - T_\infty^{(L)})_+) > \lambda\right)$$

for all continuity points λ of the right hand side, and all $\theta_1 \ldots \theta_H \in \mathbb{R}$ by, e.g., Theorem 13.2.2 in Whitt (2002). This proves the proposition. □

If one-dimensional weak convergence suffices in Proposition 2.2.3, the condition (2.29) can be replaced by a weaker uniform boundedness condition. Proposition 2.2.5 below works as a crucial piece in the proof of the main theorem in Chapter 4. The proof is essentially different from that of Proposition 2.2.3 and is entirely independent of Theorem 2.2.1.

Proposition 2.2.5. *Under the assumptions of Theorem 2.2.1, we allow the limiting case $\beta = 0$ as well. Let $A \in \mathcal{E}$, $0 < \mu(A) < \infty$, be a uniform set for T. Let $A_0 = A$, $A_n = A^c \cap \{\varphi = n\}$,*

$n \geq 1$, *where* φ *is the first entrance time of* A. *Suppose that*

$$\frac{1}{\mu(\varphi \leq n)} \sum_{k=1}^{n} \widehat{T}^k \mathbf{1}_{A_k} \text{ is uniformly bounded on } A\,. \tag{2.33}$$

Suppose that the function f *is supported by* A. *Define a probability measure on* E *by* $\mu_n(\cdot) = \mu(\cdot \cap \{\varphi \leq n\})/\mu(\{\varphi \leq n\})$. *Then under* μ_n,

$$\frac{S_n(f)}{a_n} \Rightarrow \mu(f)\Gamma(1+\beta)M_\beta(1-V_\beta) \quad \text{in } \mathbb{R}\,,$$

where V_β *is a random variable independent of the Mittag-Leffler process* M_β, *with*

$$P(V_\beta \leq x) = x^{1-\beta}, \quad 0 \leq x \leq 1\,. \tag{2.34}$$

Proof. We first claim that

$$\frac{S_n(\mathbf{1}_A)}{a_n} \Rightarrow \mu(A)\Gamma(1+\beta)M_\beta(1-V_\beta) \quad \text{in } \mathbb{R} \tag{2.35}$$

with respect to μ_n, and try to replace $\mathbf{1}_A$ by a more general function f thereafter.

Because of (2.11) and the fact that $M_\beta(t)$ is a self-similar process with self-similarity exponent β, the moments of $M_\beta(1-V_\beta)$ are given by

$$EM_\beta(1-V_\beta)^r = E(1-V_\beta)^{r\beta} EM_\beta(1)^r = r!\frac{\Gamma(2-\beta)}{\Gamma(r\beta+2-\beta)}\,.$$

Recall the fact that given the moments of any order, the Mittag-Leffler laws can be uniquely determined (see e.g., Bingham (1971)). A simple application of the Carleman sufficient condition proves that the laws of $M_\beta(1-V_\beta)$ can also be uniquely determined by their moments. From these observations, (2.35) follows if we can show that

$$\int_E \left(\frac{S_n(\mathbf{1}_A)}{a_n}\right)^r d\mu_n \to \big(\mu(A)\Gamma(1+\beta)\big)^r r!\frac{\Gamma(2-\beta)}{\Gamma(r\beta+2-\beta)}\,, \quad \text{for every } r = 1, 2, \dots$$

From now, we will repeatedly use the Karamata's Tauberian theorem for power series (see e.g., Corollary 1.7.3 in Bingham et al. (1987)).

First, we claim that

$$\sum_{n=1}^{\infty} \left(\int_E \binom{S_n(\mathbf{1}_A)}{r} d\mu \right) e^{-\lambda n} \sim \frac{1}{(r-1)!} \frac{\mu(A)}{\lambda} \sum_{n=1}^{\infty} \left(\int_A S_n(\mathbf{1}_A)^{r-1} d\mu_A \right) e^{-\lambda n} \quad \text{as } \lambda \downarrow 0\,, \tag{2.36}$$

where $\mu_A(\cdot) = \mu(\cdot \cap A)/\mu(A)$.

For the proof, the following identity is needed:

$$\binom{S_n(\mathbf{1}_A)}{r} = \sum_{k=1}^{n} \left(\mathbf{1}_A \binom{S_{n-k}(\mathbf{1}_A)}{r-1} \right) \circ T^k, \quad r = 1, 2, \dots .$$

As $\lambda \downarrow 0$, we have

$$\sum_{n=1}^{\infty} \left(\int_E \binom{S_n(\mathbf{1}_A)}{r} d\mu \right) e^{-\lambda n} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \left(\int_E \left(\mathbf{1}_A \binom{S_{n-k}(\mathbf{1}_A)}{r-1} \right) \circ T^k d\mu \right) e^{-\lambda n}$$

$$\sim \frac{\mu(A)}{\lambda} \sum_{n=1}^{\infty} \left(\int_A \binom{S_n(\mathbf{1}_A)}{r-1} d\mu_A \right) e^{-\lambda n}.$$

It is elementary to show that

$$\int_A \binom{S_n(\mathbf{1}_A)}{r-1} d\mu_A \sim \frac{1}{(r-1)!} \int_A S_n(\mathbf{1}_A)^{r-1} d\mu_A \quad n \to \infty \,,$$

which completes (2.36).

We already know from the proof of Theorem 9.1 in Thaler and Zweimüller (2006) (or Aaronson (1981)) that

$$\int_A S_n(\mathbf{1}_A)^{r-1} d\mu_A \sim \big(\mu(A)\Gamma(1+\beta)\big)^{r-1} E M_\beta(1)^{r-1} a_n^{r-1}$$

$$= \big(\mu(A)\Gamma(1+\beta)\big)^{r-1} (r-1)! \frac{a_n^{r-1}}{\Gamma((r-1)\beta+1)} \quad \text{as } n \to \infty \,.$$

Since a_n is regularly varying with exponent β, one can set $a_n = n^\beta L(n)$ by some slowly varying function L. Then, from the Karamata's Tauberian theorem,

$$\sum_{n=1}^{\infty} \left(\int_A S_n(\mathbf{1}_A)^{r-1} d\mu_A \right) e^{-\lambda n} \sim (r-1)! (\mu(A)\Gamma(1+\beta))^{r-1} \frac{1}{\lambda^{(r-1)\beta+1}} L(\lambda^{-1})^{r-1} \quad \text{as } \lambda \downarrow 0 \,. \tag{2.37}$$

Consequently, from (2.36) and (2.37),

$$\sum_{n=1}^{\infty} \left(\int_E \binom{S_n(\mathbf{1}_A)}{r} d\mu \right) e^{-\lambda n} \sim \mu(A)^r \Gamma(1+\beta)^{r-1} \frac{1}{\lambda^{r\beta+2-\beta}} L(\lambda^{-1})^{r-1} \quad \text{as } \lambda \downarrow 0 \,.$$

Since $\int_E \binom{S_n(\mathbf{1}_A)}{r} d\mu$ is nondecreasing in n and $r\beta + 2 - \beta > 0$, one more application of the Karamata's Tauberian theorem yields

$$\int_E \binom{S_n(\mathbf{1}_A)}{r} d\mu \sim \frac{\mu(A)^r \Gamma(1+\beta)^{r-1}}{\Gamma(r\beta + 2 - \beta)} n a_n^{r-1} \quad \text{as } n \to \infty\,.$$

It is not difficult to justify

$$\int_E \binom{S_n(\mathbf{1}_A)}{r} d\mu \sim \frac{1}{r!} \int_E S_n(\mathbf{1}_A)^r d\mu\,.$$

Therefore, we get

$$\int_E \left(\frac{S_n(\mathbf{1}_A)}{a_n}\right)^r d\mu \sim \mu(A)^r r! \frac{\Gamma(1+\beta)^{r-1}}{\Gamma(r\beta + 2 - \beta)} \frac{n}{a_n} \quad \text{as } n \to \infty\,. \tag{2.38}$$

Thus we get, from (2.7) and (2.8),

$$\begin{aligned} \int_E \left(\frac{S_n(\mathbf{1}_A)}{a_n}\right)^r d\mu_n &= \frac{1}{\mu(\varphi \le n)} \int_E \left(\frac{S_n(\mathbf{1}_A)}{a_n}\right)^r d\mu \\ &\to \mu(A)^r r! \frac{\Gamma(2-\beta)\Gamma(1+\beta)^r}{\Gamma(r\beta + 2 - \beta)} \quad \text{as } n \to \infty\,, \end{aligned}$$

which completes (2.35).

Next, the indicator function $\mathbf{1}_A$ must be replaced by f. To this end, it suffices to show that

$$\mu_n\left(\left| \frac{S_n(f)}{S_n(\mathbf{1}_A)} - \frac{\mu(f)}{\mu(A)} \right| > \epsilon \right) \to 0 \quad \text{for every } \epsilon > 0\,. \tag{2.39}$$

Indeed, if the above is true, the Slutsky theorem gives

$$\left(\frac{S_n(\mathbf{1}_A)}{a_n}, \frac{S_n(f)}{S_n(\mathbf{1}_A)} \right) \Rightarrow \left(\mu(A)\Gamma(1+\beta)M_\beta(1-V_\beta), \frac{\mu(f)}{\mu(A)} \right)$$

with respect to μ_n. Here, the convergence of the ratio $S_n(f)/S_n(\mathbf{1}_A)$ is obtained by the Hopf's ergodic theorem (sometimes also called a ratio ergodic theorem; see Theorem 2.2.5 in Aaronson (1997)). Applying the continuous mapping theorem, we get

$$\frac{S_n(f)}{a_n} \Rightarrow \mu(f)\Gamma(1+\beta)M_\beta(1-V_\beta) \quad \text{in } \mathbb{R}.$$

Since $\mu(A) < \infty$, it is enough to verify

$$\mu_n\left(A^c \cap \left\{\left|\frac{S_n(f)}{S_n(\mathbf{1}_A)} - \frac{\mu(f)}{\mu(A)}\right| > \epsilon\right\}\right) \to 0 \quad \text{for every } \epsilon > 0\,.$$

Denote

$$K_n = \left\{\left|\frac{f + S_n(f)}{1 + S_n(\mathbf{1}_A)} - \frac{\mu(f)}{\mu(A)}\right| > \epsilon\right\}\,.$$

Noting that f is supported by A, we obtain

$$\mu\left(A^c \cap \{\varphi \le n\} \cap \left\{\left|\frac{S_n(f)}{S_n(\mathbf{1}_A)} - \frac{\mu(f)}{\mu(A)}\right| > \epsilon\right\}\right) = \sum_{m=1}^{n} \mu\big(A_m \cap T^{-m} K_{n-m}\}\big)\,.$$

Thus, for an arbitrary constant $\delta \in (0,1)$, one can proceed as follows.

$$\mu_n\left(A^c \cap \left\{\left|\frac{S_n(f)}{S_n(\mathbf{1}_A)} - \frac{\mu(f)}{\mu(A)}\right| > \epsilon\right\}\right)$$

$$\begin{aligned}
&\le \frac{1}{\mu(\varphi \le n)} \sum_{m=1}^{\lceil (1-\delta)n \rceil} \mu\big(A_m \cap T^{-m} K_{n-m}\big) + \frac{1}{\mu(\varphi \le n)} \sum_{m=\lceil (1-\delta)n \rceil + 1}^{n} \mu(\varphi = m) \\
&= \int_A \frac{1}{\mu(\varphi \le n)} \sum_{m=1}^{\lceil (1-\delta)n \rceil} \widehat{T}^m \mathbf{1}_{A_m} \cdot \mathbf{1}_{K_{n-m}} d\mu + 1 - \frac{\mu(\varphi \le \lceil (1-\delta)n \rceil)}{\mu(\varphi \le n)} \\
&\le \int_A \frac{1}{\mu(\varphi \le \lceil (1-\delta)n \rceil)} \sum_{m=1}^{\lceil (1-\delta)n \rceil} \widehat{T}^m \mathbf{1}_{A_m} \sup_{n - \lceil (1-\delta)n \rceil \le i \le n} \mathbf{1}_{K_i} d\mu + 1 - \frac{\mu(\varphi \le \lceil (1-\delta)n \rceil)}{\mu(\varphi \le n)}\,.
\end{aligned}$$

Because of (2.33), $\mu(\varphi \le \lceil (1-\delta)n \rceil)^{-1} \sum_{m=1}^{\lceil (1-\delta)n \rceil} \widehat{T}^m \mathbf{1}_{A_m}$ is uniformly bounded on A; further, the Hopf's ergodic theorem yields

$$\sup_{n - \lceil (1-\delta)n \rceil \le i \le \ldots n} \mathbf{1}_{K_i} \to 0 \quad \text{as } n \to \infty \quad \text{a.e. on } A\,.$$

Applying the dominated convergence theorem, we conclude

$$\limsup_{n \to \infty} \mu_n\left(A^c \cap \left\{\left|\frac{S_n(f)}{S_n(\mathbf{1}_A)} - \frac{\mu(f)}{\mu(A)}\right| > \epsilon\right\}\right) \le 1 - (1-\delta)^{1-\beta}\,.$$

Letting $\delta \downarrow 0$ on the right hand side, we get (2.39). $\square$

Remark 2.2.6. In Proposition 2.2.5, we assumed that T is conservative, ergodic, measure preserving, and pointwise dual ergodic with return sequence (a_n). However, a careful inspection of Theorem 9.1 in Thaler and Zweimüller (2006) reveals the following. Suppose that a measurable map T defined on $(E, \mathcal{E}, \mu)$ is measure preserving and satisfies

$$\frac{1}{a_n} \sum_{k=1}^{n} \widehat{T}^k \mathbf{1}_A \to \mu(A) \quad \text{uniformly, a.e. on } A$$

(this condition does make sense because $\widehat{T}$ is well-defined as long as T is measure preserving). We assume neither conservativity nor ergodicity for the operator T. However, relation (2.38) still follows. This observation plays an important role in the proof of the main theorem in Chapter 4 (see particularly Lemma 4.4.3).

We will next discuss ergodic distributional convergence of the partial maxima

$$M_{\lfloor nt \rfloor}(f)(x) = \max_{1 \le k \le \lfloor nt \rfloor} |f \circ T^k(x)| \,, \quad x \in E, \ t \ge 0 \,.$$

By convention, we set $\max_{m \le k \le n} a_k = 0$ whenever $m > n$. The proposition below will be repeatedly applied in the proof of the main theorem in Chapter 5. For the sake of notational convenience in Chapter 5, we now replace the exponent β with $1 - \beta$. As a consequence, a normalizing sequence (a_n) of pointwise dual ergodicity is assumed to be regularly varying with exponent $1 - \beta$ for some $0 < \beta < 1$.

Proposition 2.2.7. *Let T be a conservative ergodic and measure preserving map on a σ-finite infinite measure space $(E, \mathcal{E}, \mu)$. We assume that T is a pointwise dual ergodic map with normalizing sequence (a_n) that is regularly varying with exponent $1 - \beta$ for some $0 < \beta < 1$. Let $A \in \mathcal{E}$, $0 < \mu(A) < \infty$, be a uniform set for T. Let $A_0 = A$, $A_n = A^c \cap \{\varphi = n\}$, $n \ge 1$, where φ is the first entrance time of A. Suppose that*

$$\frac{\widehat{T}^n \mathbf{1}_{A_n}}{\mu(A_n)} \to K \quad \textit{uniformly, a.e. on } A \,. \tag{2.40}$$

where $K : E \to \mathbb{R}_+$ is a measurable function. Define a probability measure on E by $\mu_n(\cdot) = \mu(\cdot \cap \{\varphi \le n\}) / \mu(\{\varphi \le n\})$. Let $f : E \to \mathbb{R}$ be a measurable function supported by the set A such that $|f(x)| < L$ a.e. on A for some $L > 0$.

Let $0 \le t_1 < \cdots < t_d$, $d \ge 1$. Then

$$\mu_n \circ \Big(\big(M_{\lfloor n t_i \rfloor}(f) \big)_{i=1}^d \Big)^{-1} \Rightarrow \big(\eta \times P' \big) \circ \Big(\big(M_\infty(f) \mathbf{1}_{\{V_\beta \le t_i\}} \big)_{i=1}^d \Big)^{-1} \quad \textit{in } \mathbb{R}_+^d \,,$$

where $M_\infty(f)(x) = \sup_{k\geq 1} |f \circ T^k(x)|$, $x \in E$ *and* $\eta(\cdot) = \int_\cdot K(x)\mu(dx)$ *is a probability measure on* E, *and* V_β *is a random variable defined on a probability space* $(\Omega', \mathcal{F}', P')$ *with* $P'(V_\beta \leq x) = x^\beta$, $0 < x \leq 1$.

Proof. Since T preserves measure μ, for the duration of the proof we may and will modify the definition of $M_n(f)$ to $M_n(f) = \max_{0\leq k\leq n-1} |f \circ T^k|$. Fix $\lambda > 0$ and $\theta_i > 0$, $i = 1, \dots, d$. Let $\bigvee_{i=1}^d a_i = \max_{1\leq i\leq d} a_i$. Since f is supported by A, as $n \to \infty$, we have

$$\mu_n\left(\bigvee_{i=1}^d \theta_i M_{\lfloor nt_i\rfloor}(f) > \lambda\right) \sim \mu(\varphi \leq n)^{-1} \sum_{m=1}^n \mu\left(A_m \cap T^{-m}\left\{\bigvee_{i=1}^d \theta_i M_{(\lfloor nt_i\rfloor - m)_+}(f) > \lambda\right\}\right)$$

$$= \int_A \frac{1}{\mu(\varphi \leq n)} \sum_{m=1}^n \widehat{T}^m \mathbf{1}_{A_m} \cdot \mathbf{1}\left(\bigvee_{i=1}^d \theta_i M_{(\lfloor nt_i\rfloor - m)_+}(f) > \lambda\right) d\mu\,.$$

It follows from (2.40) that

$$\int_A \frac{1}{\mu(\varphi \leq n)} \sum_{m=1}^n \widehat{T}^m \mathbf{1}_{A_m} \cdot \mathbf{1}\left(\bigvee_{i=1}^d \theta_i M_{(\lfloor nt_i\rfloor - m)_+}(f) > \lambda\right) d\mu \tag{2.41}$$

$$\sim \sum_{m=1}^n \eta\left(\bigvee_{i=1}^d \theta_i M_{(\lfloor nt_i\rfloor - m)_+}(f) > \lambda\right) p_n(m)\,,$$

where $p_n(m) = \mu(A_m)/\sum_{j=1}^n \mu(A_j)$, $m = 1, \dots, n$, is a probability mass function. Let T_n be a random variable with probability mass function $(p_n(m),\, m = 1, \dots, n)$, independent of $M_{\lfloor n\cdot\rfloor}(f)$. We may declare that T_n is defined on some probability space $(\Omega_n, \mathcal{F}_n, P_n)$. Then the right hand side of (2.41) becomes

$$\int_A P_n\left(\bigvee_{i=1}^d \theta_i M_{(\lfloor nt_i\rfloor - T_n)_+}(f) > \lambda\right) d\eta\,.$$

We note that T_n/n converges in law to V_β. Indeed, as $n \to \infty$,

$$P_n\left(\frac{T_n}{n} \leq x\right) = \sum_{m=1}^{\lfloor nx\rfloor} p_n(m) \sim \frac{\mu(\varphi \leq \lfloor nx\rfloor)}{\mu(\varphi \leq n)} \to x^\beta = P'\big(V_\beta \leq x\big)$$

for all $0 < x \leq 1$. Since V_β is a non-degenerate random variable, we conclude

$$\int_A P_n\left(\bigvee_{i=1}^d \theta_i M_{(\lfloor nt_i\rfloor - T_n)_+}(f) > \lambda\right) d\eta \to \big(\eta \times P'\big)\left(M_\infty(f) \bigvee_{i=1}^d \theta_i \mathbf{1}_{\{V_\beta \leq t_i\}} > \lambda\right)\,.$$

Note that $M_\infty(f) < \infty$ a.e. on A because f is bounded almost everywhere on A. $\square$

CHAPTER 3

FUNCTIONAL CENTRAL LIMIT THEOREM FOR HEAVY TAILED STATIONARY INFINITELY DIVISIBLE PROCESSES GENERATED BY CONSERVATIVE FLOWS

This chapter states the functional central limit theorems of the process (1.3). After describing our setup in Section 3.1, Section 3.2 introduces the limiting symmetric α-stable (henceforth, SαS) self-similar processes with stationary increments and discuss its properties. In Section 3.3, we present the statement of the functional central limit theorem and several examples. The proof of the theorem uses several distributional ergodic-theoretical results we have presented and proved in Section 2.2. Finally, the proof of the main theorem is completed in Section 3.4.

3.1 The setup

We consider infinitely divisible processes of the form

$$X_n = \int_E f_n(x)dM(x), \quad n = 1, 2, \dots , \tag{3.1}$$

where M is an infinitely divisible random measure on a measurable space $(E, \mathcal{E})$, and the functions f_n, $n = 1, 2, \dots$ are deterministic functions of the form

$$f_n(x) = f \circ T^n(x) = f\big(T^n x\big), \; x \in E, \; n = 1, 2, \dots , \tag{3.2}$$

where $f : E \to \mathbb{R}$ is a measurable function, and $T : E \to E$ a measurable map. The (independently scattered) infinitely divisible random measure M is assumed to be a homogeneous symmetric infinitely divisible random measure without a Gaussian component, with control measure μ and local Lévy measure ρ. That is, μ is a σ-finite measure on E, which we will assume to be infinite. Further, ρ is a symmetric Lévy measure on $\mathbb{R}$, and for every $A \in \mathcal{E}$ with $\mu(A) < \infty$, $M(A)$ is a (symmetric) infinitely divisible random variable such that

$$Ee^{iuM(A)} = \exp\left\{-\mu(A)\int_{\mathbb{R}} \big(1-\cos(ux)\big)\,\rho(dx)\right\} \qquad u \in \mathbb{R}.$$

Our basic assumption is the heaviness of the marginal tail of the process $\mathbf{X} = (X_1, X_2, \dots)$. We will assume that the local Lévy measure ρ has a regularly varying tail with index $-\alpha$, $0 < \alpha < 2$. That is,

$$\rho(\cdot, \infty) \in RV_{-\alpha} \text{ at infinity.} \tag{3.3}$$

We will impose an extra assumption on the lower tail of the local Lévy measure: for some $p_0 < 2$

$$x^{p_0}\rho(x, \infty) \to 0 \text{ as } x \to 0. \tag{3.4}$$

We will assume that the measurable map T preserves the control measure μ. We intend to relate the ergodic-theoretical properties of the map T to the dependence properties of the process $\mathbf{X}$ and, subsequently, to the kind of central limit theorem the process satisfies. The major "player" in that sense is the assumption that the map T is conservative. This property has already been observed to be related to long memory in the process $\mathbf{X}$; see e.g. Samorodnitsky (2004) and Roy (2008). In fact, throughout, T is assumed to be an ergodic conservative measure preserving map on an infinite σ-finite measure space $\big(E, \mathcal{E}, \mu\big)$. We will assume that the operator T has a Darling-Kac set A, with $0 < \mu(A) < \infty$, (recall (2.4)), and that the normalizing sequence (a_n) is regularly varying with exponent $\beta \in (0, 1)$. We will add an extra assumption on the set A; there exists a measurable function $K : E \to \mathbb{R}_+$ such that, in the notation of (2.29),

$$\frac{\widehat{T}^n \mathbf{1}_{A_n}}{\mu(A_n)} \to K \quad \text{uniformly, a.e. on } A. \tag{3.5}$$

This condition is an extension of the property shared by certain operators T, the so-called Markov shifts (see Chapter 4 in Aaronson (1997) and Remark 2.2.4), to a more general class of operators. See examples 3.3.5 and 3.3.6 below.

Let $f : E \to \mathbb{R}$ be a measurable function with the following integrability properties:

$$f \in \begin{cases} L^{1\vee p}(\mu) \text{ for some } \ p > p_0 & \text{if } 0 < \alpha < 1 \\ L^{\infty}(\mu) & \text{if } \alpha = 1 \\ L^2(\mu) & \text{if } 1 < \alpha < 2 \end{cases} . \tag{3.6}$$

We will, further, assume that

$$\mu(f) = \int_E f(s)\, \mu(ds) \neq 0 \,, \tag{3.7}$$

and that f is supported by the Darling-Kac set A.

We consider again a stochastic process $\mathbf{X} = \big(X_1, X_2, \ldots\big)$ of the form (3.1) - (3.2). The integral is well defined under the condition

$$\int_E \int_{\mathbb{R}} \min\big(1, x^2 f_n(s)^2\big)\, \rho(dx)\, \mu(ds) < \infty \,.$$

It is not difficult to verify that this condition holds due to the assumptions on the Lévy measure ρ and the integrability conditions (3.6) on f. Once the process $\mathbf{X}$ is well defined, it is, automatically, a symmetric and infinitely divisible process, without a Gaussian component. The function level Lévy measure of the process $\mathbf{X}$ is given by

$$\kappa = (\rho \times \mu) \circ K^{-1} \,,$$

where $K(x,s) = x\big(f_1(s), f_2(s), \ldots\big)$, $s \in E$, $x \in \mathbb{R}$. Since the Lévy measure κ is invariant under the left shift θ and, hence, the process $\mathbf{X}$ is stationary. For details see Rajput and Rosiński (1989).

Let $H : \mathbb{R} \times E \to \mathbb{R}$ be defined by $H(x,s) = xf(s)$. Then the assumptions on the Lévy measure ρ and the integrability conditions (3.6) on f imply that

$$(\rho \times \mu) \circ H^{-1}(\lambda, \infty) \sim \Big(\int_E |f(s)|^{\alpha}\, \mu(ds)\Big) \rho(\lambda, \infty)$$

as $\lambda \to \infty$. It follows that the marginal tail of the process itself is the same:

$$P(X_n > \lambda) \sim \Big(\int_E |f(s)|^{\alpha}\, \mu(ds)\Big) \rho(\lambda, \infty)$$

as $\lambda \to \infty$; see Rosiński and Samorodnitsky (1993). In particular, the marginal distributions of the process $\mathbf{X}$ are in the domain of attraction of a SαS law.

The assumptions on the Lévy measure ρ and the operator T lead to a rather satisfying picture, in which the kind of the central limit theorem that holds for the process $\mathbf{X}$ depends both on the marginal tails of the process and on the length of memory in it. Both are clearly parametrized by the exponents $\alpha \in (0,2)$ and $\beta \in (0,1)$, respectively.

We proceed, first, with a description of the limiting process we will eventually obtain.

3.2 The limiting process

In this section, we will introduce a class of self-similar SαS processes with stationary increments. These processes will later appear as weak limits in the central limit theorem. We will see this process is an extension (to a wider range of parameters) of a class recently introduced by Dombry and Guillotin-Plantard (2009). Before introducing this process, we need to recall the Mittag-Leffler process. For $0 < \beta < 1$, let $\big(S_\beta(t), t \geq 0\big)$ be a β-stable subordinator, satisfying $Ee^{-\theta S_\beta(t)} = \exp\{-t\theta^\beta\}$ for $\theta \geq 0$ and $t \geq 0$. Define the Mittag-Leffler process by

$$M_\beta(t) = S_\beta^{\leftarrow}(t) = \inf\big\{u \geq 0 : \ S_\beta(u) \geq t\big\}, \ t \geq 0 \,. \tag{3.8}$$

We will frequently use the Laplace transform of $M_\beta(t)$

$$E\exp\{\theta M_\beta(t)\} = \sum_{n=0}^{\infty} \frac{(\theta t^\beta)^n}{\Gamma(1+n\beta)}, \quad \theta \in \mathbb{R} \,. \tag{3.9}$$

See Section 2.2 for several properties of the Mittag-Leffler processes.

We are now ready to introduce the new class of self-similar SαS processes with stationary increments. Let $0 < \alpha < 2$ and $0 < \beta < 1$, and let $(\Omega', \mathcal{F}', P')$ be a probability space. We define

$$Y_{\alpha,\beta}(t) = \int_{\Omega' \times [0,\infty)} M_\beta\big((t-x)_+, \omega'\big) dZ_{\alpha,\beta}(\omega', x), \quad t \geq 0, \tag{3.10}$$

where $Z_{\alpha,\beta}$ is a SαS random measure on $\Omega' \times [0,\infty)$ with control measure $P' \times \nu_\beta$, with ν_β a measure on $[0,\infty)$ given by $\nu_\beta(dx) = (1-\beta)x^{-\beta}\, dx$, $x > 0$. Here M_β is a Mittag-Leffler process defined on $(\Omega', \mathcal{F}', P')$. The random measure $Z_{\alpha,\beta}$ itself and, hence, also the process $Y_{\alpha,\beta}$, are defined on some generic probability space $(\Omega, \mathcal{F}, P)$. We refer the reader to Samorodnitsky and Taqqu (1994) for more information on integrals with respect to stable random measures.

In Theorem 3.2.1 below we prove that the process $\big(Y_{\alpha,\beta}(t),\, t \geq 0\big)$ is a well defined self-similar SαS processes with stationary increments. We call it *the β-Mittag-Leffler* (or β-ML) *fractional SαS motion*.

Theorem 3.2.1. *The β-ML fractional SαS motion is a well defined self-similar SαS processes with stationary increments. It is also self-similar with exponent of self-similarity $H = \beta + (1-\beta)/\alpha$.*

Proof. By the monotonicity of the process M_β we have, for any $t \geq 0$,

$$\int_{[0,\infty)} \int_{\Omega'} M_\beta((t-x)_+, \omega')^\alpha P'(d\omega) \nu_\beta(dx) \leq t^\beta E' M_\beta(t)^\alpha < \infty\,,$$

which proves that the process $\big(Y_{\alpha,\beta}(t),\, t \geq 0\big)$ is well defined. Further, by the β-self-similarity of the process M_β, we have for any $k \geq 1$, $t_1 \ldots t_k \geq 0$, and $c > 0$, for all real $\theta_1, \ldots, \theta_k$,

$$E \exp\left\{ i \sum_{j=1}^k \theta_j Y_{\alpha,\beta}(ct_j) \right\} = \exp\left\{ -\int_0^\infty E' \Big| \sum_{j=1}^k \theta_j M_\beta((ct_j - x)_+) \Big|^\alpha (1-\beta) x^{-\beta} dx \right\}$$

$$= \exp\left\{ -\int_0^\infty E' \Big| \sum_{j=1}^k \theta_j c^H M_\beta((t_j - y)_+) \Big|^\alpha (1-\beta) y^{-\beta} dy \right\} = E \exp\left\{ i \sum_{j=1}^k \theta_j c^H Y_{\alpha,\beta}(t_j) \right\},$$

which shows the H-self-similarity of the β-ML fractional SαS motion.

For the proof of stationary increment property, it suffices to check that

$$E\exp\left\{i\sum_{j=1}^k \theta_j\big(Y_{\alpha,\beta}(t_j+s)-Y_{\alpha,\beta}(s)\big)\right\} = E\exp\left\{i\sum_{j=1}^k \theta_j Y_{\alpha,\beta}(t_j)\right\}$$

for all $k\geq 1$, $t_1\dots t_k\geq 0$, $s\geq 0$, and $\theta_1\dots\theta_k\in\mathbb{R}$. This is equivalent to verifying the equality in

$$\int_0^\infty E'\Big|\sum_{j=1}^k \theta_j\{M_\beta((t_j+s-x)_+)-M_\beta((s-x)_+)\}\Big|^\alpha x^{-\beta}dx$$

$$=\int_0^\infty E'\Big|\sum_{j=1}^k \theta_j M_\beta((t_j-x)_+)\Big|^\alpha x^{-\beta}dx\,.$$

Changing variable by $r=s-x$ in the left hand side and rearranging the terms shows that we need to check the equality in

$$\int_0^s E'\Big|\sum_{j=1}^k \theta_j(M_\beta(t_j+r)-M_\beta(r))\Big|^\alpha (s-r)^{-\beta}dr \tag{3.11}$$

$$=\int_0^\infty E'\Big|\sum_{j=1}^k \theta_j M_\beta((t_j-x)_+)\Big|^\alpha (x^{-\beta}-(s+x)^{-\beta})dx.$$

Let $\delta_r=S_\beta\big(M_\beta(r)\big)-r$ be the overshoot of the level $r>0$ by the β-stable subordinator $\big(S_\beta(t),t\geq 0\big)$ related to $\big(M_\beta(t),t\geq 0\big)$ by (3.8). The law of δ_r is known to be given by

$$P(\delta_r\in dx)=\frac{\sin\beta\pi}{\pi}r^\beta(r+x)^{-1}x^{-\beta}\,dx,\ x>0\,; \tag{3.12}$$

see e.g. Exercise 5.6 in Kyprianou (2006). Further, by the strong Markov property of the stable subordinator we have

$$\big(M_\beta(t+r)-M_\beta(r),\,t\geq 0\big)\stackrel{d}{=}\big(M_\beta((t-\delta_r)_+),\,t\geq 0\big)\,,$$

with the understanding that M_β and δ_r in the right hand side are independent. We conclude that

$$\int_0^s E'|\sum_{j=1}^k \theta_j(M_\beta(t_j+r)-M_\beta(r))|^\alpha (s-r)^{-\beta}dr \tag{3.13}$$

$$=\frac{\sin\beta\pi}{\pi}\int_0^\infty\int_0^s E'|\sum_{j=1}^k \theta_j M_\beta((t_j-x)_+)|^\alpha r^\beta(r+x)^{-1}x^{-\beta}(s-r)^{-\beta}drdx.$$

Using the integration formula

$$\int_0^1 \left(\frac{t}{1-t}\right)^{\beta} \frac{1}{t+y}\, dt = \frac{\pi}{\sin \beta\pi}\left[1 - \left(\frac{y}{1+y}\right)^{\beta}\right], \; y > 0\,,$$

given on p. 338 of Gradshteyn and Ryzhik (1994), shows that (3.13) is equivalent to (3.11). This completes the proof. $\square$

Recall that, when $0 < \beta \leq 1/2$, the Mittag-Leffler process of (3.8) is distributionally equivalent to the local time at zero of a symmetric stable Lévy process with index of stability $\hat{\beta} = (1-\beta)^{-1}$. Specifically, let $\bigl(W_{\hat{\beta}}(t),\, t \geq 0\bigr)$ be a symmetric $\hat{\beta}$-stable Lévy process, such that $Ee^{irW_{\hat{\beta}}(t)} = \exp\{-t|r|^{\hat{\beta}}\}$ for $r \in \mathbb{R}$ and $t \geq 0$. This process has a jointly continuous local time process, $L_t(x)$, $t \geq 0$, $x \in \mathbb{R}$; see e.g. Getoor and Kesten (1972). Then

$$\bigl(M_\beta(t),\, t \geq 0\bigr) \stackrel{d}{=} \bigl(c_\beta L_t(0),\, t \geq 0\bigr) \tag{3.14}$$

for some $c_\beta > 0$; see Section 11.1.1 in Marcus and Rosen (2006). Therefore, in the range $0 < \beta \leq 1/2$, the β-ML fractional SαS motion (3.10) can be represented in law as

$$Y_{\alpha,\beta}(t) = c_\beta \int_{\Omega' \times [0,\infty)} L_{(t-x)_+}\bigl(0, \omega'\bigr)\, dZ_{\alpha,\beta}(\omega', x), \quad t \geq 0,$$

where $\bigl(L_t(x)\bigr)$ is the local time of a symmetric $\hat{\beta}$-stable Lévy process defined on $(\Omega', \mathcal{F}', P')$. Recall also the $\hat{\beta}$-stable local time fractional SαS motion introduced in Dombry and Guillotin-Plantard (2009) (see also Cohen and Samorodnitsky (2006)). That process can be defined by

$$\hat{Y}_{\alpha,\beta}(t) = \int_{\Omega' \times \mathbb{R}} L_t\bigl(x, \omega'\bigr)\, d\hat{Z}_\alpha(\omega', x), \quad t \geq 0,$$

where $\hat{Z}_\alpha$ is a SαS random measure on $\Omega' \times \mathbb{R}$ with control measure $P' \times$ Leb. We claim that, in fact, if $0 < \beta \leq 1/2$,

$$\bigl(Y_{\alpha,\beta}(t)\;\, t \geq 0\bigr) \stackrel{d}{=} c_\beta^{(1)}\bigl(\hat{Y}_{\alpha,\beta}(t)\;\, t \geq 0\bigr)\,, \tag{3.15}$$

for some multiplicative constant $c_\beta^{(1)}$. Therefore, one can view the ML fractional SαS motion as an extension of the $\hat{\beta}$-stable local time fractional SαS motion from the range

$1 < \hat{\beta} \leq 2$ to the range $1 < \hat{\beta} < \infty$. It is interesting to note that the central limit theorem in Section 3.3 is of a very different type from the random walk in random scenery situation of Cohen and Samorodnitsky (2006) and Dombry and Guillotin-Plantard (2009).

To check (3.15), let

$$H_x = \inf\{t \geq 0 : \ W_{\hat{\beta}}(t) = x\}, \ x \in \mathbb{R} .$$

Since $1 < \hat{\beta} \leq 2$, H_x is a.s. finite for any $x \in \mathbb{R}$; see e.g. Remark 43.12 in Sato (1999). Further, by the strong Markov property, for every $x \in \mathbb{R}$, the conditional law of $\big(L_{H_x+t}(x), \, t \geq 0\big)$ given $\mathcal{F}'_{H_x}$, coincides a.s. with the law of $\big(L_t(0), \, t \geq 0\big)$. We conclude that for any $k \geq 1$, $t_1 \ldots t_k \geq 0$, and real $\theta_1, \ldots, \theta_k$,

$$-\log E \exp\Big\{\sum_{j=1}^{k} \theta_j \hat{Y}_{\alpha,\beta}(t_j)\Big\} = \int_{\mathbb{R}} E'\Big|\sum_{j=1}^{k} \theta_j L_{t_j}(x)\Big|^{\alpha} dx$$

$$= \int_{\mathbb{R}} \int_0^{\infty} E'\Big|\sum_{j=1}^{k} \theta_j L_{(t_j-y)_+}(0)\Big|^{\alpha} F_x(dy)\, dx \, ,$$

where F_x is the law of H_x. Using the obvious fact that $H_x \overset{d}{=} |x|^{\hat{\beta}} H_1$, an easy calculation shows that the mixture $\int_{\mathbb{R}} F_x \, dx$ is, up to a multiplicative constant, equal to the measure ν_β in (3.10). Therefore, for some constant $c_\beta^{(1)}$,

$$-\log E \exp\Big\{\sum_{j=1}^{k} \theta_j c_\beta^{(1)} \hat{Y}_{\alpha,\beta}(t_j)\Big\} = -\log E \exp\Big\{\sum_{j=1}^{k} \theta_j Y_{\alpha,\beta}(t_j)\Big\} ,$$

and (3.15) follows.

Remark 3.2.2. It is interesting to observe that, for a fixed $0 < \alpha < 2$, the range of the exponent of self-similarity $H = \beta + (1-\beta)/\alpha$ of the β-ML fractional SαS motion, as β varies between 0 and 1, is a proper subset of the feasible range of the exponent of self-similarity of stationary increment self-similar SαS processes, which is $0 < H \leq \max(1, 1/\alpha)$; see Samorodnitsky and Taqqu (1994).

It was shown in Dombry and Guillotin-Plantard (2009) that the stable local time fractional SαS motion is Hölder continuous. We extend this statement to the ML fractional SαS motion.

Theorem 3.2.3. *The β-ML fractional $S\alpha S$ motion satisfies, with probability 1,*

$$\sup_{0\le s<t\le 1/2} \frac{\big|Y_{\alpha,\beta}(t)-Y_{\alpha,\beta}(s)\big|}{(t-s)^\beta\big|\log(t-s)\big|^{1-\beta}} < \infty$$

if $0<\alpha<1$, *and*

$$\sup_{0\le s<t\le 1/2} \frac{\big|Y_{\alpha,\beta}(t)-Y_{\alpha,\beta}(s)\big|}{(t-s)^\beta\big|\log(t-s)\big|^{3/2-\beta}} < \infty$$

if $1\le\alpha<2$.

Proof. The statement of the theorem follows from Lemma 3.2.4 and the argument in Theorem 5.1 in Cohen and Samorodnitsky (2006); see also Theorem 1.5 in Dombry and Guillotin-Plantard (2009). □

The next lemma establishes Hölder continuity of the Mittag-Leffler process (3.8). The statement might be known, but we could not find a reference, so we present a simple argument. In the case $0<\beta\le 1/2$ (most of) the statement is in Theorem 2.1 in Ehm (1981), through the relation with the local time (3.14).

Lemma 3.2.4. *For $B>0$ let*

$$K = \sup_{0\le s<t<s+1/2\le B} \frac{\big|M_\beta(t)-M_\beta(s)\big|}{(t-s)^\beta\big|\log(t-s)\big|^{1-\beta}}\,.$$

Then K is an a.s. finite random variable with all finite moments.

Proof. Because of the self-similarity of the Mittag-Leffler process it is enough to consider $B=1/2$. In the course of the proof we will use the notation $c(\beta)$ for a finite positive constant that may depend on β, and that may change from one appearance to another. Recall the lower tail estimate of a positive β-stable random variable:

$$P\big(S_\beta(1)\le\theta\big)\le \exp\big\{-c(\beta)\theta^{-\beta/(1-\beta)}\big\},\ 0<\theta\le 1\,; \tag{3.16}$$

see Zolotarev (1986). Let $\lambda\ge 1$. We have

$$P(K>\lambda)\le\sum_{n=1}^\infty P\Big(\sup_{\substack{0\le s<t\le 1/2\\ 2^{-(n+1)}\le t-s\le 2^{-n}}} M_\beta(t)-M_\beta(s) > c(\beta)\lambda n^{1-\beta}2^{-n\beta}\Big) := \sum_{n=1}^\infty q_n(\lambda)\,.$$

For $n = 1, 2, \ldots$ we use the following decomposition:

$$q_n(\lambda) \leq P\big(S_\beta(\lambda \log n) \leq 1/2\big)$$

$$+P\Big[\text{for some } 0 < t \leq \lambda \log n,\ S_\beta\Big(t + c(\beta)\lambda n^{1-\beta} 2^{-n\beta}\Big) - S_\beta(t) \leq 2^{-n}\Big] := q_n^{(1)}(\lambda) + q_n^{(2)}(\lambda)\,.$$

Using (3.16) and self-similarity of the stable subordinator, we obtain

$$\sum_{n=1}^{\infty} q_n^{(1)}(\lambda) \leq c(\beta)^{-1} \exp\big\{-c(\beta)\lambda^{1/(1-\beta)}\big\}\,.$$

On the other hand,

$$q_n^{(2)}(\lambda) \leq P\Big(S_\beta\Big(2^{-1}(i+1)c(\beta)\lambda n^{1-\beta} 2^{-n\beta}\Big) - S_\beta\Big(2^{-1} i c(\beta)\lambda n^{1-\beta} 2^{-n\beta}\Big) \leq 2^{-n},$$

$$\text{some } i = 0, \ldots, K_n\Big)\,,$$

with $K_n \leq 2c(\beta)^{-1} n^{\beta-1} 2^{n\beta} \log n$. Switching to the complements, and using once again (3.16) together with the independence of the increments and self-similarity of the stable subordinator, we conclude, after some straightforward calculus, that for all $\lambda \geq \lambda(\beta) \in (0, \infty)$,

$$\sum_{n=1}^{\infty} q_n^{(2)}(\lambda) \leq c(\beta)^{-1} \exp\big\{-c(\beta)\lambda^{1/(1-\beta)}\big\}\,.$$

The resulting bound on the tail probability $P(K > \lambda)$ is sufficient for the statement of the lemma. $\square$

Recall that the only self-similar Gaussian process with stationary increments is the Fractional Brownian motion (FBM), whose law is, apart from the scale, uniquely determined by the self-similarity parameter $H \in (0, 1)$; see Samorodnitsky and Taqqu (1994). This parameter of self-similarity also determines the dependence properties of the increment process of the FBM, the so-called Fractional Gaussian noise, with the case $H > 1/2$ regarded as the long memory case. In contrast, the self-similarity parameter almost never determines the dependence properties of the increment processes of stable self-similar processes with stationary increments; see Samorodnitsky (2006).

Therefore, it is interesting and important to discuss the memory properties of the increment process

$$V_n^{(\alpha,\beta)} = Y_{\alpha,\beta}(n+1) - Y_{\alpha,\beta}(n), \quad n = 0, 1, 2, \ldots, .$$

We refer the reader to Rosiński (1995) and Samorodnitsky (2005) for some of the notions used in the statement of the following theorem.

Theorem 3.2.5. *The stationary process* $\big(V_n^{(\alpha,\beta)}\big)$ *is generated by a conservative null flow and is mixing.*

Proof. Note that the increment process has the integral representation

$$V_n^{(\alpha,\beta)} = \int_{\Omega' \times [0,\infty)} \big(M_\beta\big((n+1-x)_+, \omega'\big) - M_\beta\big((n-x)_+, \omega'\big)\big)\, dZ_{\alpha,\beta}(\omega', x),\ n = 0, 1, 2, \ldots .$$

Since for every $x > 0$, on a set of P' probability 1, by the strong Markov property of the stable subordinator we have

$$\limsup_{n\to\infty} M_\beta\big((n+1-x)_+\big) - M_\beta\big((n-x)_+\big) > 0\,,$$

we see that

$$\sum_{n=1}^{\infty} \big(M_\beta\big((n+1-x)_+, \omega'\big) - M_\beta\big((n-x)_+, \omega'\big)\big)^\alpha = \infty \quad P' \times \nu_\beta \text{ a.e..}$$

By Corollary 4.2 in Rosiński (1995) we conclude that the increment process is generated by a conservative flow.

It remains to prove that the increment process is mixing, since mixing implies ergodicity which, in turns, implies that the increment process is generated by a null flow; see Samorodnitsky (2005). By Theorem 5 of Rosiński and Żak (1996), it is enough to show that for every $\epsilon > 0$,

$$(P' \times \nu_\beta)\{(\omega', x) : M_\beta((1-x)_+, \omega') > \epsilon,\ M_\beta((n+1-x)_+, \omega') - M_\beta((n-x)_+, \omega') > \epsilon\}$$

$$\to 0 \quad \text{as } n \to \infty\,.$$

However, an obvious upper bound on the expression in the left hand side is

$$\int_0^1 P'\big(M_\beta(n+1-x) - M_\beta(n-x) > \epsilon\big)(1-\beta)x^{-\beta}\,dx$$

$$= \int_0^1 P'\big(M_\beta((1-\delta_{n-x})_+) > \epsilon\big)(1-\beta)x^{-\beta}\,dx\,,$$

where for $r > 0$, δ_r is a random variable, independent of the Mittag-Leffler process, with the distribution given by (3.12). Since δ_r converges weakly to infinity as $r \to \infty$, by the dominated convergence theorem, the above expression converges to zero as $n \to \infty$. □

Remark 3.2.6. Two extreme cases deserve mentioning. A formal substitution of $\beta = 0$ into (3.9) leads to a well-defined process $M_0(0) = 0$ and $M_0(t) = E$, the same standard exponential random variable for all $t > 0$. It can, however, be used in (3.10). It is elementary to see that the resulting SαS process $Y_{\alpha,0}$ is, in fact, a SαS Lévy motion.

On the other hand, a formal substitution of $\beta = 1$ into (3.9) leads to the degenerate process $M_1(t) = t$ for all $t \geq 0$ (which can be viewed as the inverse of the degenerate 1-stable subordinator $S_1(t) = t$ for $t \geq 0$.) Once again, this process can be used in (3.10), if one interprets the measure ν_β as the unit point mass at the origin. The resulting SαS process $Y_{\alpha,1}$ is now the degenerate process $Y_{\alpha,1}(t) = tY_{\alpha,1}(1)$ for all $t \geq 0$, where $Y_{\alpha,1}(1)$ is a SαS random variable.

Both limiting cases, $Y_{\alpha,0}$ and $Y_{\alpha,1}$, are processes of a very different nature from the β-ML fractional SαS motion with $0 < \beta < 1$.

3.3 Central Limit Theorem Associated with Conservative Flows

In this section we state and discuss a functional central limit theorem for stationary infinitely divisible processes described in Section 3.1. We will first determine the normalizing sequence (c_n) in the functional central limit theorem below. Let

$\rho^{\leftarrow}(y) = \inf\{x \geq 0 : \rho(x,\infty) \leq y\}$, $y > 0$ be the left continuous inverse of the tail of the local Lévy measure. The regular variation of the tail implies that $\rho^{\leftarrow} \in RV_{1/\alpha}$ at infinity. Define

$$c_n = \Gamma(1+\beta)\, C_\alpha^{-1/\alpha}\, a_n\, \rho^{\leftarrow}(1/w_n),\ n = 1, 2, \dots\ , \tag{3.17}$$

where C_α is the α-stable tail constant given by

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x\, dx\right)^{-1} = \begin{cases} (1-\alpha)/\Gamma(2-\alpha)\cos(\pi\alpha/2) & \text{if } \alpha \neq 1, \\ 2/\pi & \text{if } \alpha = 1; \end{cases} \tag{3.18}$$

see Samorodnitsky and Taqqu (1994). Moreover, (a_n) is the normalizing sequence in the Darling-Kac property (2.4) (or, equivalently, in the pointwise dual ergodicity property (2.2)), and (w_n) is the wandering rate sequence for the set A (related to the sequence (a_n) via (2.8)). It follows immediately that

$$c_n \in RV_{\beta+(1-\beta)/\alpha}.$$

We will see that under the conditions of that theorem we have the asymptotic relation

$$\rho\,(c_n/a_n,\infty) \sim C_\alpha\big(C_{\alpha,\beta}/\Gamma(1+\beta)\big)^\alpha |\mu(f)|^\alpha a_n^\alpha \left(\int_E |\sum_{k=1}^n f \circ T^k(x)|^\alpha \mu(dx)\right)^{-1} \quad \text{as } n \to \infty\,, \tag{3.19}$$

with

$$C_{\alpha,\beta} = \Gamma(1+\beta)\Big((1-\beta)B(1-\beta, 1+\alpha\beta)E\left(M_\beta(1)\right)^\alpha\Big)^{1/\alpha}.$$

Here B is the standard beta function, and M_β the Mittag-Leffler process defined in (3.8). The following is our functional central limit theorem.

Theorem 3.3.1. *Let T be an ergodic conservative measure preserving map on an infinite σ-finite measure space $(E, \mathcal{E}, \mu)$, possessing a Darling-Kac set A whose normalizing sequence (a_n) is regularly varying with exponent $\beta \in (0,1)$. Assume that (3.5) holds. Let M be a symmetric homogeneous infinitely divisible random measure on $(E, \mathcal{E})$ with control measure μ and local Lévy measure ρ, satisfying the regular variation with index $-\alpha$, $0 < \alpha < 2$ at infinity* *condition (3.3). Assume, further, that (3.4) holds for some $p_0 < 2$.*

Let f be a measurable function supported by A and satisfying (3.6) and (3.7). If $1 < \alpha < 2$, assume further that either

(i) A is a uniform set for $|f|$, or

(ii) f is bounded.

Then the stationary infinitely divisible stochastic process $\mathbf{X} = \big(X_1, X_2, \ldots\big)$ given by (3.1) and (3.2) satisfies

$$\frac{1}{c_n}\sum_{k=1}^{\lceil n\cdot\rceil} X_k \Rightarrow |\mu(f)| Y_{\alpha,\beta} \quad \text{in } D[0,\infty)\,, \tag{3.20}$$

where (c_n) is defined by (3.17), and $\{Y_{\alpha,\beta}\}$ is the β-Mittag-Leffler fractional SαS motion defined by (3.10).

Remark 3.3.2. The type of the limiting process obtained in Theorem 3.3.1 is an indication of the long memory in the process $\mathbf{X}$. On the other hand, the Darling-Kac assumption (2.4) and the duality relation(2.1) imply that

$$\frac{1}{a_n}\sum_{k=1}^{n}\mu(A\cap T^{-k}A) = \frac{1}{a_n}\sum_{k=1}^{n}\int_E \mathbf{1}_A\cdot\mathbf{1}_A\circ T^k\,d\mu = \int_A \frac{1}{a_n}\sum_{k=1}^{n}\widehat{T}^k\mathbf{1}_A\,d\mu \to \mu(A)^2 \in (0,\infty)$$

as $n\to\infty$. Since $a_n = o(n)$, and f is supported by A, we see that for every $\epsilon > 0$,

$$\frac{1}{n}\sum_{k=1}^{n}\mu\big\{x\in E : |f(x)|>\epsilon, |f\circ T^k(x)|>\epsilon\big\} \le \frac{1}{n}\sum_{k=1}^{n}\mu(A\cap T^{-k}A)\to 0,$$

and it follows immediately, e.g. from Theorem 2 in Rosiński and Żak (1997), that the process $\mathbf{X}$ is ergodic.

Under certain additional assumptions on the map T, one can check that the process $\mathbf{X}$ is, in fact, mixing. We skip the details. See, however, examples 3.3.5 and 3.3.6 below.

Remark 3.3.3. The statement of Theorem 3.3.1 makes sense in the limiting cases $\beta = 0$ and $\beta = 1$ of Remark 3.2.6 (in the case $\beta = 1$ the constant $C_{\alpha,1}$ needs to be interpreted as $C_\alpha^{1/\alpha}$). Most of the argument in the proof of Theorem 3.3.1 automatically works in these cases. The limiting processes would then turn out to be, correspondingly, a SαS Lévy motion and the straight line process; see Remark 3.2.6. This case $\beta = 0$ corresponds to short memory in the process $\mathbf{X}$, while the case $\beta = 1$ corresponds to extremely long memory.

Remark 3.3.4. When $0 < \alpha < 1$, the argument we will use in the proof of Theorem 3.3.1 can be used to establish a "positive" version of the theorem. Specifically, assume now that the local Lévy measure ρ is concentrated on $(0, \infty)$, and that the function f is nonnegative. Then

$$\frac{1}{c_n} \sum_{k=1}^{\lceil n \cdot \rceil} X_k \Rightarrow \mu(f) Y_{\alpha,\beta}^{+} \quad \text{in } D[0,\infty)\,, \tag{3.21}$$

where $\{Y_{\alpha,\beta}^{+}\}$ is a positive β-Mittag-Leffler fractional α-stable motion defined as in (3.10), but with SαS random measure $Z_{\alpha,\beta}$ replaced by a positive α-stable random measure with the same control measure.

We finish this section with two examples of different situations where Theorem 3.3.1 applies. The first example is close to the heart of a probabilist.

Example 3.3.5. Consider an irreducible null recurrent Markov chain with state space $\mathbb{Z}$ and transition matrix $P = (p_{ij})$. Let $\{\pi_j, j \in \mathbb{Z}\}$ be the unique invariant measure of the Markov chain that satisfies $\pi_0 = 1$. We define a σ-finite measure on $(E, \mathcal{E}) = \big(\mathbb{Z}^{\mathbb{N}}, \mathcal{B}(\mathbb{Z}^{\mathbb{N}})\big)$ by

$$\mu(\cdot) = \sum_{i \in \mathbb{Z}} \pi_i P_i(\cdot)\,,$$

with the usual notation of $P_i(\cdot)$ being the probability law of the Markov chain starting in state $i \in \mathbb{Z}$. Since $\sum_j \pi_j = \infty$, μ is an infinite measure.

Let $T : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}^{\mathbb{N}}$ be the left shift map $T(x_0, x_1, \dots) = (x_1, x_2, \dots)$ for $\{x_k, k = 0, 1, \dots\} \in \mathbb{Z}^{\mathbb{N}}$. Obviously, T preserves the measure μ. Since the Markov chain is irreducible and null recurrent, the flow $\{T^n\}$ is conservative and ergodic; see Harris and Robbins (1953).

Consider the set $A = \big\{x \in \mathbb{Z}^{\mathbb{N}} : x_0 = 0\big\}$ and the corresponding first entrance time

$\varphi(x) = \min\{n \geq 1 : x_n = 0\}$, $x \in \mathbb{Z}^{\mathbb{N}}$. Assume that

$$\sum_{k=1}^{n} P_0(\varphi \geq k) \in RV_{1-\beta}$$

for some $\beta \in (0,1)$. Since $\mu(\varphi = k) = P_0(\varphi \geq k)$ for $k \geq 1$ (see Lemma 3.3 in Resnick et al. (2000)), we see that $\mu(\varphi \leq n) \in RV_{1-\beta}$ and, hence, by (2.7), the wandering rates (w_n) have the same property,

$$w_n \in RV_{1-\beta}\,. \tag{3.22}$$

In this example,

$$\widehat{T}^k \mathbf{1}_A(x) = P_0(x_k = 0), \;\; \text{constant for } x \in A;$$

see Section 4.5 in Aaronson (1997). In particular, the set A is a Darling-Kac set, and by (3.22) and (2.8), we see that the corresponding normalizing sequence (a_n) is regularly varying with exponent β. The assumption (3.5) is easily seen to hold in this example. Indeed, applying the explicit expression for the dual operator given on p. 156 in Aaronson (1997) to the function

$$f(x_0, x_1, \ldots) = \mathbf{1}\big(x_j \neq 0,\, j = 0, \ldots, n-1,\, x_n = 0\big)\,,$$

we see that

$$\widehat{T}^n \mathbf{1}_{A_n}\big(x_0, x_1, \ldots\big) = \mathbf{1}\big(x_0 = 0\big) \sum_{i_0 \neq 0} \pi_{i_0} \sum_{i_1 \neq 0} p_{i_0 i_1} \ldots \sum_{i_{n-1} \neq 0} p_{i_{n-2} i_{n-1}} p_{i_{n-1} 0}$$

is constant on A and vanishes outside of A. Therefore, the ratio in (3.5) is identically equal to 1 on A.

We conclude that Theorem 3.3.1 applies in this case if we choose any measurable function f supported by A and satisfying the conditions of the theorem.

It is easy to see that the stationary infinitely divisible process $\mathbf{X}$ in this example is mixing. Indeed, by Theorem 5 of Rosiński and Żak (1996) it is enough to check that

$$\mu\big\{x : |f(x)| > \epsilon,\, |f \circ T^n(x)| > \epsilon\big\} \to 0$$

for every $\epsilon > 0$. However, since f vanishes outside of A, null recurrence implies that as $n \to \infty$,

$$\mu\big\{x : |f(x)| > \epsilon,\ |f \circ T^n(x)| > \epsilon\big\} \leq \mu(A \cap T^{-n}A) = P_0(x_n = 0) \to 0.$$

The next example is less familiar to probabilists, but is well known to ergodic theorists.

Example 3.3.6. We start with a construction of the so-called *AFN-system*, studied in, e.g., Zweimüller (2000) and Thaler and Zweimüller (2006). Let E be the union of a finite family of disjoint bounded open intervals in $\mathbb{R}$ and let $\mathcal{E}$ be the Borel σ-field on E. Let λ be the one-dimensional Lebesgue measure.

Let ξ be a (possibly, infinite) collection of nonempty disjoint open subintervals (of the intervals in E) such that $\lambda\big(E \setminus \bigcup_{Z \in \xi} Z\big) = 0$. Let $T : E \to E$ be a map that is twice differentiable on (each interval of) E. We assume that T is strictly monotone on each $Z \in \xi$.

The map T is further assumed to satisfy the following three conditions, (A), (F), and (N), (giving rise to the name AFN-system).

(A) *Adler's condition*:

$$T''/(T')^2 \ \text{ is bounded on } \bigcup_{Z \in \xi} Z\,.$$

(F) *Finite image condition*:

$$\text{the collection } T\xi = \{TZ : Z \in \xi\} \ \text{ is finite.}$$

(N) *A possibility of non-uniform expansion*: there exists a finite subset $\zeta \subseteq \xi$ such that each $Z \in \zeta$ has *an indifferent fixed point* x_Z as one of its end points. That is,

$$\lim_{x \to x_Z, x \in Z} Tx = x_Z \quad \text{and} \quad \lim_{x \to x_Z, x \in Z} T'x = 1.$$

Moreover, we suppose, for each $Z \in \zeta$,

$$\text{either } T' \text{ decreases on } (-\infty, x_Z) \cap Z, \ \text{ or } T' \text{ increases on } (x_Z, \infty) \cap Z\,,$$

depending on whether x_Z is the left endpoint or the right endpoint of Z. Finally, we assume that T is uniformly expanding away from $\{x_Z : Z \in \zeta\}$, i.e. for each $\epsilon > 0$, there is $\rho(\epsilon) > 1$ such that

$$|T'| \geq \rho(\epsilon) \text{ on } E \setminus \bigcup_{Z\in\zeta} \Big((x_Z - \epsilon, x_Z + \epsilon) \cap Z \Big).$$

If the conditions (A), (F), and (N) are satisfied, the triplet (E, T, ξ) is called an AFN-system, and the map T is called an AFN-map. If T is also conservative and ergodic with respect to λ, and the collection ζ is nonempty, then the AFN-map T is said to be *basic*; we will assume this property in the sequel. Finally, we will assume that T admits *nice expansions* at the indifferent fixed points. That is, for every $Z \in \zeta$ there is $0 < \beta_Z < 1$ such that

$$Tx = x + a_Z |x - x_Z|^{1/\beta_Z + 1} + o\big(|x - x_Z|^{1/\beta_Z + 1}\big) \quad \text{as } x \to x_Z \text{ in } Z, \tag{3.23}$$

for some $a_Z \neq 0$.

It is shown in Zweimüller (2000) that every basic AFN-map has an infinite invariant measure $\mu \ll \lambda$ with the density given by $d\mu/d\lambda(x) = h_0(x)G(x)$, $x \in E$, where

$$G(x) = \begin{cases} (x - x_Z)\big(x - (T|_Z)^{-1}(x)\big)^{-1} & \text{if } x \in Z \in \zeta, \\ 1 & \text{if } x \in E \setminus \bigcup_{Z\in\zeta} Z\,, \end{cases}$$

and h_0 is a function of bounded variation bounded away from both 0 and infinity. We view T as a conservative ergodic measure-preserving map on the infinite measure space $(E, \mathcal{E}, \mu)$.

An example of a basic AFN-map is Boole's transformation placed on $E = (0, 1/2) \cup (1/2, 1)$, defined by

$$T(x) = \frac{x(1-x)}{1 - x - x^2},\ x \in (0, 1/2),\ \ T(x) = 1 - T(1-x),\ x \in (1/2, 1)\,.$$

It admits nice expansions at the indifferent fixed points $x_Z = 0$ and $x_Z = 1$ with $\beta_Z = 1/2$ in both cases. The invariant measure μ satisfies

$$\frac{d\mu}{d\lambda}(x) = \frac{1}{x^2} + \frac{1}{(1-x)^2},\ x \in E\,.$$

See Thaler (2001).

Let T be a basic AFN-map. We put

$$A = E \setminus \bigcup_{Z \in \zeta} \Big((x_Z - \epsilon, x_Z + \epsilon) \cap Z \Big)$$

for some $\epsilon > 0$ small enough so that the set A is non-empty. Since $\lambda(\partial A) = 0$ and A is bounded away from the indifferent fixed points $\{x_Z : Z \in \zeta\}$, it follows from Corollary 3 of Zweimüller (2000) that A is a Darling-Kac set. Moreover, the corresponding normalizing sequence (a_n) is regularly varying with exponent $\beta = \min_{Z \in \zeta} \beta_Z$ in the notation of (3.23); see Theorems 3 and 4 in Zweimüller (2000). The assumption (3.5) also holds; see (2.6) in Thaler and Zweimüller (2006).

Once again, Theorem 3.3.1 applies if we choose any measurable function f supported by A and satisfying the conditions of Theorem 3.3.1. Note that, by Theorem 9 in Zweimüller (2000), Riemann integrability of $|f|$ on A suffices for the uniformity of the set A for $|f|$.

The stationary infinitely divisible process $\mathbf{X}$ in this example is also mixing. Indeed, the basic AFN-map T is *exact*, i.e. the σ-field $\cap_{n=1}^{\infty} T^{-n}\mathcal{B}$ is trivial; see e.g. p. 1522 in Zweimüller (2000) . The exactness of T implies that

$$\mu(A \cap T^{-n}A) = \int_A \widehat{T}^n \mathbf{1}_A \, d\mu \to 0$$

as $n \to \infty$; see p. 12 in Thaler (2001). Now mixing of the process $\mathbf{X}$ follows from the fact that f is supported by A, as in Example 3.3.5.

3.4 Proof of the Main Theorem

In this section we prove Theorem 3.3.1. We start with several preliminary results. The first lemma explains the asymptotic relation (3.19).

Lemma 3.4.1. *Under the assumptions of Proposition 2.2.3, assume, additionally, that the set A supporting f is a Darling-Kac set. Let $0 < \alpha < 2$. If $1 < \alpha < 2$, assume, additionally, that $f \in L^2(\mu)$, and that either*
(i) A is a uniform set for $|f|$, or
(ii) f is bounded.
Then

$$\left(\int_E |S_n(f)|^\alpha d\mu \right)^{1/\alpha} \sim |\mu(f)|\, C_{\alpha,\beta}\, a_n w_n^{1/\alpha} \quad \text{as } n \to \infty, \tag{3.24}$$

and (3.19) holds.

Proof. It is an elementary calculation to check that (3.24) implies (3.19), so in the sequel we concentrate on checking (3.24). It follows from (2.7) and the fact that f is supported by A, that

$$\left(\int_E |S_n(f)|^\alpha d\mu \right)^{1/\alpha} = a_n \big(\mu(\varphi \le n)\big)^{1/\alpha} A_n^{(\alpha)} \sim a_n w_n^{1/\alpha} A_n^{(\alpha)} ,$$

where $A_n^{(\alpha)} = (\int_E |S_n(f)/a_n|^\alpha d\mu_n)^{1/\alpha}$. Therefore, proving (3.24) reduces to checking that

$$A_n^{(\alpha)} \to |\mu(f)|\, C_{\alpha,\beta} \text{ as } n \to \infty. \tag{3.25}$$

If $\alpha = 1$ and f is nonnegative, then this follows by direct calculation, using the definition of $C_{\alpha,\beta}$. If f is not necessarily nonnegative, we can use the obvious bound $-S_n(|f|) \le S_n(f) \le S_n(|f|)$ together with the so-called Pratt lemma; see Pratt (1960), or Problem 16.4 (a) in Billingsley (1996).

It remains to consider the case $\alpha \in (0,1) \cup (1,2)$. Proposition 2.2.3 shows that $\big(A_n^{(\alpha)}\big)$ is the sequence of the α-norms of a weakly converging sequence, and the expression in the right hand side of (3.25) is easily seen to be the α-norm of the weak limit. Therefore, our statement will follow once we show that this weakly convergent sequence is uniformly integrable, which we proceed now to do.

Suppose first that $0 < \alpha < 1$. Recalling the relation (2.8) and the fact that T preserves measure μ, we see that

$$\sup_{n\geq 1} \int_E \left| \frac{S_n(f)}{a_n} \right| d\mu_n = \sup_{n\geq 1} \frac{1}{a_n \mu(\varphi \leq n)} \int_E |S_n(f)| d\mu \tag{3.26}$$

$$\leq \sup_{n\geq 1} \frac{n}{a_n \mu(\varphi \leq n)} \int_E |f| d\mu < \infty \, ,$$

which proves uniformly integrability in this case.

Finally, we consider the case $1 < \alpha < 2$, when it is sufficient to prove that

$$\sup_{n\geq 1} \int_E \left(\frac{S_n(f)}{a_n} \right)^2 d\mu_n < \infty. \tag{3.27}$$

Under the assumption (i), since f is supported by A, we can use the duality relation (2.1) to write

$$\int_E S_n(f)^2 d\mu = n \int_E f^2 d\mu + \sum_{k=1}^{n} \sum_{l=1, k\neq l}^{n} \int_E f \circ T^k f \circ T^l d\mu$$

$$= n \int_E f^2 d\mu + 2 \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} \int_A \widehat{T}^j f \cdot f d\mu \, ,$$

so that

$$\int_E \left(\frac{S_n(f)}{a_n} \right)^2 d\mu_n \leq \frac{n}{a_n^2 \mu(\varphi \leq n)} \int_E f^2 d\mu + \frac{2}{a_n^2 \mu(\varphi \leq n)} \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} \int_A \widehat{T}^j |f| \cdot |f| d\mu.$$

Clearly, $n / \big(a_n^2 \mu(\varphi \leq n) \big) \to 0$. Further, since A is uniform for $|f|$,

$$\frac{1}{a_n^2 \mu(\varphi \leq n)} \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} \int_A \widehat{T}^j |f| \cdot |f| d\mu \leq \frac{n}{a_n \mu(\varphi \leq n)} \int_A \frac{1}{a_n} \sum_{j=1}^{n} \widehat{T}^j |f| \cdot |f| d\mu$$

$$\sim \mu(|f|)^2 \frac{n}{a_n \mu(\varphi \leq n)} \, .$$

Using (2.8), we see that (3.27) follows. On the other hand, under the assumption (ii), the ratio $S_n(f)/S_n(\mathbf{1}_A)$ is bounded, hence for some finite $C > 0$,

$$\sup_{n\geq 1} \int_E \left(\frac{S_n(f)}{a_n} \right)^2 d\mu_n \leq C \sup_{n\geq 1} \int_E \left(\frac{S_n(\mathbf{1}_A)}{a_n} \right)^2 d\mu_n.$$

However, the Darling-Kac property of A means that it is uniform for $\mathbf{1}_A$, and so we are, once again, under the assumption (i). $\square$

In preparation for the proof of Theorem 3.3.1, we introduce a useful decomposition of the process $\mathbf{X}$ given in (3.1). We begin by decomposing the local Lévy measure ρ into a sum of two parts, corresponding to "large jumps" and "small jumps". Let

$$\rho_1(\cdot) = \rho\big(\cdot \cap \{|x| > 1\}\big),$$

$$\rho_2(\cdot) = \rho\big(\cdot \cap \{|x| \leq 1\}\big),$$

and let $M_1,\ M_2$ be independent homogeneous symmetric infinitely divisible random measures, without a Gaussian component, with the same control measure μ and local Lévy measures $\rho_1,\ \rho_2$ accordingly. Under the integrability assumptions (3.6), the stochastic processes $X_n^{(i)} = \int_E f \circ T^n(x) dM_i(x)$, $n = 1, 2, \ldots$, for $i = 1, 2$, are independent stationary infinitely divisible processes, and $X_n = X_n^{(1)} + X_n^{(2)}$, $n = 1, 2, \ldots$.

Our final lemma shows that, from the point of view of the central limit behavior in the case $0 < \alpha < 1$, the contribution of the process $\big(X_n^{(2)}\big)$, corresponding to the "small jumps", is negligible.

Lemma 3.4.2. *If* $0 < \alpha < 1$, *then*

$$\frac{1}{c_n} \sum_{k=1}^{n} X_k^{(2)} \xrightarrow{p} 0\,.$$

Proof. By Chebyshev's inequality, for any $\epsilon > 0$,

$$P\left(|\sum_{k=1}^{n} X_k^{(2)}| > \epsilon c_n\right) \leq \frac{n}{\epsilon c_n} E|X_1^{(2)}| \to 0$$

(since $c_n \in RV_{\beta+(1-\beta)/\alpha}$ implies $n/c_n \to 0$ in the case $0 < \alpha < 1$) as long as the expectation $E|X_1^{(2)}|$ is finite. Since for every $p_1 > p_0$ in (3.4) and $p_1 \geq 1$,

$$\int_E \int_{\mathbb{R}} |xf(s)| \mathbf{1}\big(|xf(s)| > 1\big)\, \rho_2(dx)\, \mu(ds) \leq \int_{-1}^{1} |x|^{p_1}\, \rho(dx) \int_E |f(s)|^{p_1}\, \mu(ds)\,,$$

the expectation is finite because, by (3.6), we can find p_1 as above such that $\int_E |f|^{p_1}\, d\mu < \infty$. ☐

Proof of Theorem 3.3.1. We start with proving the finite dimensional weak convergence, for which it is enough to show the convergence

$$\frac{1}{c_n}\sum_{h=1}^{H}\theta_h\sum_{k=1}^{\lceil nt_h\rceil}X_k \Rightarrow |\mu(f)|\sum_{h=1}^{H}\theta_h Y_{\alpha,\beta}(t_h)$$

for all $H \geq 1$, $0 \leq t_1 < \cdots < t_H$, and $\theta_1 \ldots \theta_H \in \mathbb{R}$. Conditions for weak convergence of infinitely divisible random variables (see e.g. Theorem 15.14 in Kallenberg (2002)) simplify in this one-dimensional symmetric case to

$$\int_E \left(\frac{1}{c_n}\sum_{h=1}^{H}\theta_h S_{\lceil nt_h\rceil}(f)\right)^2 \int_0^{rc_n/|\sum\theta_h S_{\lceil nt_h\rceil}(f)|} x\rho(x,\infty)\,dx\,d\mu \tag{3.28}$$

$$\to \frac{r^{2-\alpha}C_\alpha}{2-\alpha}\,|\mu(f)|^\alpha \int_{[0,\infty)}\int_{\Omega'}\left|\sum_{h=1}^{H}\theta_h M_\beta((t_h-x)_+,\omega')\right|^\alpha P'(d\omega')\nu_\beta(dx)$$

and

$$\int_E \rho\left(rc_n|\sum_{h=1}^{H}\theta_h S_{\lceil nt_h\rceil}(f)|^{-1},\infty\right)d\mu \tag{3.29}$$

$$\to r^{-\alpha}C_\alpha\,|\mu(f)|^\alpha \int_{[0,\infty)}\int_{\Omega'}\left|\sum_{h=1}^{H}\theta_h M_\beta((t_h-x)_+,\omega')\right|^\alpha P'(d\omega')\nu_\beta(dx)$$

for every $r > 0$. Fix $L \in \mathbb{N}$ with $t_H \leq L$ and $r > 0$.

Since the argument for (3.28) and the argument for (3.29) are very similar, we only prove (3.28). By Proposition 2.2.3 and Skorohod's embedding theorem, there is some probability space $(\Omega^*, \mathcal{F}^*, P^*)$ and random variables Y, Y_n, $n = 1, 2, \ldots$ defined on that space such that, for every n, the law of Y_n coincides with the law of $a_n^{-1}\sum_{h=1}^{H}\theta_h S_{\lceil nt_h\rceil}(f)$ under μ_{nL}, the law of Y coincides with the law of $\mu(f)\Gamma(1+\beta)\sum_{h=1}^{H}\theta_h M_\beta((t_h - T_\infty^{(L)})_+)$ under P', and $Y_n \to Y$ P^*-a.s.

Introduce a function

$$\psi(y) = y^{-2}\int_0^{ry} x\rho(x,\infty)\,dx,\ y > 0\,,$$

so that the expression in the left hand side of (3.28) becomes

$$\int_E \psi\left(\frac{c_n}{|\sum_{h=1}^{H}\theta_h S_{\lceil nt_h\rceil}(f)|}\right)d\mu = \mu(\varphi \leq nL)\,E^*\left[\psi\left(\frac{c_n}{a_n|Y_n|}\right)\right].$$

By Karamata's theorem (see e.g. Theorem 0.6 in Resnick (1987)),

$$\psi(y) \sim \frac{r^2}{2-\alpha}\rho(ry,\infty) \quad \text{as } y \to \infty\,,$$

so that, as $n \to \infty$,

$$\begin{aligned}
&\mu(\varphi \le nL)\psi\left(\frac{c_n}{a_n|Y_n|}\right) \qquad (3.30)\\
&\sim \frac{r^2}{2-\alpha}\mu(\varphi \le nL)|Y_n|^\alpha \rho\big(rc_n a_n^{-1},\infty\big)\\
&+\frac{r^2}{2-\alpha}\mu(\varphi \le nL)\rho\big(rc_n a_n^{-1},\infty\big)\left(\frac{\rho\big(rc_n a_n^{-1}|Y_n|^{-1},\infty\big)}{\rho\big(rc_n a_n^{-1},\infty\big)} - |Y_n|^\alpha\right).
\end{aligned}$$

By (3.19), Lemma 3.4.1 and (2.7),

$$\rho(rc_n a_n^{-1},\infty) \sim r^{-\alpha}C_\alpha\big(\Gamma(1+\beta)\big)^{-\alpha}\big(\mu(\varphi \le n)\big)^{-1} \quad \text{as } n \to \infty\,. \qquad (3.31)$$

This, together with the basic properties of regularly varying functions of a negative index (see e.g. Proposition 0.5 Resnick (1987)), shows that the second term in the right hand side of (3.30) converges to 0. Therefore,

$$\mu(\varphi \le nL)\psi\left(\frac{c_n}{a_n|Y_n|}\right) \to \frac{r^{2-\alpha}}{2-\alpha}C_\alpha L^{1-\beta}\left(\frac{|Y|}{\Gamma(1+\beta)}\right)^\alpha.$$

Integrating the limit yields

$$\begin{aligned}
E^*\left[\frac{r^{2-\alpha}}{2-\alpha}C_\alpha\, L^{1-\beta}\left(\frac{|Y|}{\Gamma(1+\beta)}\right)^\alpha\right] &= \frac{r^{2-\alpha}}{2-\alpha}C_\alpha\, L^{1-\beta}\,|\mu(f)|^\alpha E'\left[\sum_{h=1}^H \theta_h M_\beta((t_h - T_\infty^{(L)})_+)\right]^\alpha\\
&= \frac{r^{2-\alpha}C_\alpha}{2-\alpha}\,|\mu(f)|^\alpha \int_{[0,\infty)}\int_{\Omega'}\left(\sum_{h=1}^H \theta_h M_\beta((t_h - x)_+,\omega')\right)^\alpha P'(d\omega')\nu_\beta(dx),
\end{aligned}$$

which is exactly the right hand side of (3.28). Therefore, in order to complete the proof of (3.28), we only need to justify taking the limit inside the integral. For this purpose we use, once again, Pratt's lemma. We need to exhibit random variables G_n, $n = 0,1,2,\ldots$ on $\big(\Omega^*,\mathcal{F}^*,P^*\big)$ such that

$$\mu(\varphi \le nL)\psi\left(\frac{c_n}{a_n|Y_n|}\right) \le G_n \quad P^*\text{-a.s.}\,, \qquad (3.32)$$

$$G_n \to G_0 \quad P^*\text{-a.s.}\,, \qquad (3.33)$$

$$E^*G_n \to E^*G_0 \in [0,\infty)\,. \qquad (3.34)$$

We start with writing (using (3.31))

$$\mu(\varphi \le nL)\psi\left(\frac{c_n}{a_n|Y_n|}\right) \le C_1 \frac{\psi\big(c_n a_n^{-1}|Y_n|^{-1}\big)}{\psi(c_n a_n^{-1})}\mathbf{1}_{\{c_n > a_n|Y_n|\}} + C_1 \frac{\psi\big(c_n a_n^{-1}|Y_n|^{-1}\big)}{\psi(c_n a_n^{-1})}\mathbf{1}_{\{c_n \le a_n|Y_n|\}}\,,$$

where $C_1 > 0$ is a constant. Suppose first that $1 \le \alpha < 2$, and choose $0 < \xi < 2 - \alpha$. Then by the Potter bounds (see Proposition 0.8 in Resnick (1987)), for some constant $C_2 > 0$,

$$\frac{\psi\big(c_n a_n^{-1}|Y_n|^{-1}\big)}{\psi(c_n a_n^{-1})}\mathbf{1}_{\{c_n > a_n|Y_n|\}} \le C_2(|Y_n|^{\alpha-\xi} + |Y_n|^{\alpha+\xi})$$

for all n large enough. Further, since $y^2\psi(y) \to 0$ as $y \downarrow 0$, we have, for some constant $C_3 > 0$,

$$\frac{\psi\big(c_n a_n^{-1}|Y_n|^{-1}\big)}{\psi(c_n a_n^{-1})}\mathbf{1}_{\{c_n \le a_n|Y_n|\}} \le C_3\left(\frac{a_n}{c_n}\right)^2 \frac{|Y_n|^2}{\psi(c_n a_n^{-1})}\,,$$

hence, for some constant $C_4 > 0$,

$$\mu(\varphi \le nL)\psi\left(\frac{c_n}{a_n|Y_n|}\right) \le C_4\left(|Y_n|^{\alpha-\xi} + |Y_n|^{\alpha+\xi} + \left(\frac{a_n}{c_n}\right)^2 \frac{|Y_n|^2}{\psi(c_n a_n^{-1})}\right) \tag{3.35}$$

for all n (large enough) and all realizations. We take

$$G_n = C_4\left(|Y_n|^{\alpha-\xi} + |Y_n|^{\alpha+\xi} + \left(\frac{a_n}{c_n}\right)^2 \frac{|Y_n|^2}{\psi(c_n a_n^{-1})}\right) \quad n = 1, 2, \ldots,$$

$$G_0 = C_4(|Y|^{\alpha-\xi} + |Y|^{\alpha+\xi})\,.$$

Then (3.32) holds by construction, while (3.33) follows from the fact that

$$\left(\frac{a_n}{c_n}\right)^2 \frac{1}{\psi(c_n a_n^{-1})} \in RV_{(1-\beta)(1-2/\alpha)}\,,$$

and $(1-\beta)(1-2/\alpha) < 0$. Keeping this in mind, and recalling that, by (3.27) (which holds also for $\alpha = 1$ under the assumptions of the theorem), $\sup_{n\ge1} E^*Y_n^2 < \infty$, we obtain the uniform integrability implying (3.34). This proves (3.28) in the case $1 \le \alpha < 2$.

If $0 < \alpha < 1$, then Lemma 3.4.2 allows us to assume, without loss of generality, that $\rho(x : |x| \le 1) = 0$. Then ψ is bounded on $(0, 1]$, so that for some $C_5 > 0$,

$$\frac{\psi(c_n a_n^{-1}|Y_n|^{-1})}{\psi(c_n a_n^{-1})}\mathbf{1}_{\{c_n \le a_n|Y_n|\}} \le C_5 \frac{a_n}{c_n} \frac{|Y_n|}{\psi(c_n a_n^{-1})}\,,$$

and the upper bound (3.35) is replaced with

$$\mu(\varphi \le nL)\psi\left(\frac{c_n}{a_n|Y_n|}\right) \le C_6\left(|Y_n|^{\alpha-\xi} + |Y_n|^{\alpha+\xi} + \frac{a_n}{c_n}\frac{|Y_n|}{\psi(c_n a_n^{-1})}\right),$$

for some $C_6 > 0$, where we now choose $0 < \xi < 1-\alpha$. Since

$$\frac{a_n}{c_n}\frac{1}{\psi(c_n a_n^{-1})} \in RV_{(1-\beta)(1-1/\alpha)}$$

with $(1-\beta)(1-1/\alpha) < 0$ and $\sup_{n\ge 1} E^*|Y_n| < \infty$ by (3.26), an argument similar to the case $1 \le \alpha < 2$ applies here as well. A similar argument proves, in the case $0 < \alpha < 1$, the "positive" version described in Remark 3.3.4.

It remains to prove that the laws in the left hand side of (3.20) are tight in $D[0, L]$ for any fixed $L > 0$. By Theorem 13.5 of Billingsley (1999), it is enough to show that there exist $\gamma_1 > 1$, $\gamma_2 \ge 0$ and $B > 0$ such that

$$P\Big[\min\Big(\Big|\sum_{k=1}^{\lceil ns\rceil} X_k - \sum_{k=1}^{\lceil nr\rceil} X_k\Big|, \Big|\sum_{k=1}^{\lceil nt\rceil} X_k - \sum_{k=1}^{\lceil ns\rceil} X_k\Big|\Big) \ge \lambda c_n\Big] \le \frac{B}{\lambda^{\gamma_2}}(t-r)^{\gamma_1}$$

for all $0 \le r \le s \le t \le L$, $n \ge 1$ and $\lambda > 0$. We start with a simple observation that, in the case $0 < \alpha < 1$, we may assume that the function f is bounded. To see that, note that we can always write $f = f\mathbf{1}_{|f|>M} + f\mathbf{1}_{|f|\le M}$, and use the finite-dimensional convergence in (3.21) and the fact that $\mu(f\mathbf{1}_{|f|>M}) \to 0$ as $M \to \infty$.

Next, for any $0 < \alpha < 2$, if $0 < t-r < 1/n$, then the probability in the left hand side vanishes. If $X_n = X_n^{(1)} + X_n^{(2)}$, $n = 1, 2, \ldots$ be the decomposition described prior to Lemma 3.4.2. We start with the part corresponding to the "small jumps". Note that, by Lemma 3.4.2, this part is negligible if $0 < \alpha < 1$ (since we can apply the lemma to the supremum of the process). Therefore, we only consider the case $1 \le \alpha < 2$, and prove that there exist $\gamma_1 > 1$, $\gamma_2 \ge 0$ and $B > 0$ such that for all $0 \le s \le t \le L$, $n \ge 1$, $|t-s| \ge 1/n$ and $\lambda > 0$,

$$P\Big(\Big|\sum_{k=1}^{\lceil nt\rceil} X_k^{(2)} - \sum_{k=1}^{\lceil ns\rceil} X_k^{(2)}\Big| \ge \lambda c_n\Big) \le \frac{B}{\lambda^{\gamma_2}}(t-s)^{\gamma_1}. \tag{3.36}$$

Note that the Lévy-Itô decomposition yields

$$\sum_{k=1}^{\lceil nt\rceil} X_k^{(2)} - \sum_{k=1}^{\lceil ns\rceil} X_k^{(2)} \stackrel{d}{=} \int_E S_{\lceil nt\rceil-\lceil ns\rceil}(f)\, dM_2$$

$$\stackrel{d}{=} \iint_{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|\le \lambda c_n} xS_{\lceil nt\rceil-\lceil ns\rceil}(f)\, d\bar{N}_2 + \iint_{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|> \lambda c_n} xS_{\lceil nt\rceil-\lceil ns\rceil}(f)\, dN_2\,,$$

where N_2 is a Poisson random measure on $\mathbb{R}\times E$ with mean measure $\rho_2\times\mu$ and $\bar{N}_2 \equiv N_2 - \big(\rho_2\times\mu\big)$. Therefore,

$$P\Big(\Big|\sum_{k=1}^{\lceil nt\rceil} X_k^{(2)} - \sum_{k=1}^{\lceil ns\rceil} X_k^{(2)}\Big| \ge \lambda c_n\Big)$$

$$\le P\Big(\Big|\iint_{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|\le \lambda c_n} xS_{\lceil nt\rceil-\lceil ns\rceil}(f)\, d\bar{N}_2\Big| \ge \lambda c_n\Big) + P\Big(\Big|\iint_{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|> \lambda c_n} xS_{\lceil nt\rceil-\lceil ns\rceil}(f)\, dN_2\Big| > 0\Big)\,.$$

It follows from (3.4) that for some constant $C_1>0$,

$$P\Big(\Big|\iint_{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|\le \lambda c_n} xS_{\lceil nt\rceil-\lceil ns\rceil}(f)\, d\bar{N}_2\Big| \ge \lambda c_n\Big) \le \frac{1}{\lambda^2 c_n^2} E\left|\iint_{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|\le \lambda c_n} xS_{\lceil nt\rceil-\lceil ns\rceil}(f)\, d\bar{N}_2\right|^2$$

$$= \frac{1}{\lambda^2 c_n^2}\iint_{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|\le \lambda c_n} \big|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)\big|^2\,\rho_2(dx)\,d\mu$$

$$\le 4\int_E \left(\frac{S_{\lceil nt\rceil-\lceil ns\rceil}(f)}{\lambda c_n}\right)^2 \int_0^{\lambda c_n/|S_{\lceil nt\rceil-\lceil ns\rceil}(f)|} x\rho_2(x,\infty)\,dx\,d\mu$$

$$\le \frac{C_1}{\lambda^{p_0}}\frac{1}{c_n^{p_0}}\int_E |S_{\lceil nt\rceil-\lceil ns\rceil}(f)|^{p_0}\,d\mu\,.$$

Similarly, for some constant $C_2>0$,

$$P\Big(\Big|\iint_{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|> \lambda c_n} xS_{\lceil nt\rceil-\lceil ns\rceil}(f)\, dN_2\Big| > 0\Big) \le P\big(N_2\{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|>\lambda c_n\}\ge 1\big)$$

$$\le EN_2\{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|>\lambda c_n\}$$

$$= 2\int_E \rho_2\big(\lambda c_n|S_{\lceil nt\rceil-\lceil ns\rceil}(f)|^{-1},\infty\big)\,d\mu$$

$$\le \frac{C_2}{\lambda^{p_0}}\frac{1}{c_n^{p_0}}\int_E |S_{\lceil nt\rceil-\lceil ns\rceil}(f)|^{p_0}\,d\mu\,,$$

so that, in the notation of (3.25),

$$P\Big(\Big|\sum_{k=1}^{\lceil nt\rceil} X_k^{(2)} - \sum_{k=1}^{\lceil ns\rceil} X_k^{(2)}\Big| \ge \lambda c_n\Big) \le \frac{C_1+C_2}{\lambda^{p_0}}\frac{1}{c_n^{p_0}}\int_E |S_{\lceil nt\rceil-\lceil ns\rceil}(f)|^{p_0}\,d\mu$$

$$= \frac{C_1 + C_2}{\lambda^{p_0}} \frac{\mu(\varphi \le \lceil nt \rceil - \lceil ns \rceil)}{\mu(\varphi \le n)} \left(\frac{a_{\lceil nt \rceil - \lceil ns \rceil}}{a_n} \right)^{p_0} \frac{(A^{(p_0)}_{\lceil nt \rceil - \lceil ns \rceil})^{p_0}}{c_n^{p_0} \mu(\varphi \le n)^{-1} a_n^{-p_0}} .$$

It follows from (3.27) that

$$\sup_{n \ge 1, 0 \le s \le t \le L} A^{(p_0)}_{\lceil nt \rceil - \lceil ns \rceil} < \infty.$$

Next, we may, if necessary, increase p_0 in (3.4) to achieve $p_0 > \alpha$. In that case, the sequence $c_n^{p_0} \mu(\varphi \le n)^{-1} a_n^{-p_0} \in RV_{(1-\beta)(p_0/\alpha - 1)}$ diverges to infinity, so for some constant $C_3 > 0$,

$$\frac{1}{c_n^{p_0}} \int_E |S_{\lceil nt \rceil - \lceil ns \rceil}(f)|^{p_0} d\mu \le C_3 \frac{\mu(\varphi \le \lceil n(t-s) \rceil)}{\mu(\varphi \le n)} \left(\frac{a_{\lceil n(t-s) \rceil}}{a_n} \right)^{p_0} .$$

By the regular variation and the constraint $t - s \ge 1/n$, for every $0 < \eta < \min(\beta, 1-\beta)$, there is $C_4 > 0$, such that

$$\frac{\mu(\varphi \le \lceil n(t-s) \rceil)}{\mu(\varphi \le n)} \le C_4 \left(\frac{\lceil n(t-s) \rceil}{n} \right)^{1-\beta-\eta} \le 2^{1-\beta-\eta} C_4 \, (t-s)^{1-\beta-\eta} ,$$

$$\frac{a_{\lceil n(t-s) \rceil}}{a_n} \le 2^{\beta - \eta} C_4 \, (t-s)^{\beta - \eta} .$$

Therefore, for some constant $C_5 > 0$,

$$P\Big(\Big| \sum_{k=1}^{\lceil nt \rceil} X_k^{(2)} - \sum_{k=1}^{\lceil ns \rceil} X_k^{(2)} \Big| \ge \lambda c_n \Big) \le C_5 \frac{1}{\lambda^{p_0}} (t-s)^{1 + (p_0 - 1)\beta - (1+p_0)\eta} .$$

Since $p_0 > \alpha \ge 1$, we can choose $\eta > 0$ so small that $1 + (p_0 - 1)\beta - (1 + p_0)\eta > 0$. This establishes (3.36).

Next, we take up the process $\big(X_n^{(1)}\big)$. Lévy-Itô decomposition and the symmetry of the Lévy measure ρ_1 allow us to write, for any $K > 0$,

$$\frac{1}{c_n} \sum_{k=1}^{\lceil nt \rceil} X_k^{(1)} \stackrel{d}{=} \frac{1}{c_n} \sum_{k=1}^{\lceil nt \rceil} \iint_{|x f_k| \le K c_n a_n^{-1}} x f_k \, d\bar{N}_1 + \frac{1}{c_n} \sum_{k=1}^{\lceil nt \rceil} \iint_{|x f_k| > K c_n a_n^{-1}} x f_k \, dN_1$$

$$:= Z_n^{(1,K)}(t) + Z_n^{(2,K)}(t),$$

where N_1 and $\bar{N}_1$ are as above. Here we first show that or any $\epsilon > 0$,

$$\lim_{K \to \infty} \limsup_{n \to \infty} P\big(\sup_{0 \le t \le L} \big| Z_n^{(2,K)}(t) \big| \ge \epsilon \big) = 0 . \qquad (3.37)$$

Consider first the case $1 < \alpha < 2$. Choose $0 < \tau \le 2 - \alpha$, and define

$$\kappa(w) = \begin{cases} 1 & \text{if } 0 \le w < 1 \\ w^{-(\alpha+\tau)} & \text{if } w \ge 1, \end{cases}$$

$$g(w) = \big((w+1)\kappa(w)\big)^{-1}, \;\; w \ge 0\,.$$

Since $2g(w)/g(u) \ge 1$ for $0 \le u \le w$, we have

$$P\big(\sup_{0\le t\le L} \big|Z_n^{(2,K)}(t)\big| \ge \epsilon\big) \le P\left(\iint_{\mathbb{R}\times E} |x| \sum_{k=1}^{nL} |f|\circ T^k\, \mathbf{1}\big(|x||f|\circ T^k > Kc_n a_n^{-1}\big)\, dN_1 \ge \epsilon c_n\right)$$

$$= P\left(2\iint_{\mathbb{R}\times E} |x| \sum_{k=1}^{nL} |f|\circ T^k\, g\big(|f|\circ T^k\big) \frac{1}{g\big(Kc_n a_n^{-1}/|x|\big)}\, dN_1 \ge \epsilon c_n\right)$$

$$\le \frac{2}{\epsilon} c_n^{-1} E\left(\iint_{\mathbb{R}\times E} |x| \sum_{k=1}^{nL} |f|\circ T^k\, g\big(|f|\circ T^k\big) \frac{1}{g\big(Kc_n a_n^{-1}/|x|\big)}\, dN_1\right)$$

$$\le C_1\, n c_n^{-1} \int_1^\infty x\big(Kc_n a_n^{-1}/x + 1\big)\kappa\big(Kc_n a_n^{-1}/x\big)\, \rho(dx)\,,$$

where $C_1 > 0$ is another constant. It is now straightforward to check that for some constant $C_2 > 0$,

$$\limsup_{n\to\infty} P\big(\sup_{0\le t\le L} \big|Z_n^{(2,K)}(t)\big| \ge \epsilon\big) \le C_2 K^{-(\alpha-1)}\,.$$

This implies (3.37).

On the other hand, let $0 < \alpha \le 1$. Recall that we are assuming that the function f is now bounded. We have

$$P\big(\sup_{0\le t\le L} |Z_n^{(2,K)}(t)| \ge \epsilon\big) \le P\big(\max_{k=1,\dots,nL} N_1\big\{(x,s):\, |xf_k(s)| > Kc_n a_n^{-1}\big\} \ge 1\big)$$

$$\le EN_1\big\{(x,s):\, |x| \max_{k=1,\dots,nL} |f_k| > Kc_n a_n^{-1}\big\}$$

$$= 2\int_E \rho_1\left(\frac{Kc_n a_n^{-1}}{\max_{k=1,\dots,nL}|f_k|}, \infty\right) d\mu\,.$$

If we denote $\|f\| = \sup_{x\in E} |f(x)| < \infty$, then we can use once again Potter's bounds to see that for some constant $C_1 > 0$ and $0 < \xi < \alpha$,

$$\frac{\rho_1\left(Kc_n a_n^{-1}(\max_k |f_k|)^{-1}, \infty\right)}{\rho_1(c_n a_n^{-1}, \infty)} \le C_1 \left(\left(\frac{1}{K}\max_{k=1,\dots,nL} |f_k|\right)^{\alpha-\xi} + \left(\frac{1}{K}\max_{k=1,\dots,nL} |f_k|\right)^{\alpha+\xi}\right).$$

Therefore by (2.7), (3.19) and the fact that f is supported by A, for some constant $C_2 > 0$,

$$
\begin{aligned}
&P\Big(\sup_{0\le t\le L} |Z_n^{(2,K)}(t)| \ge \epsilon\Big) \\
&\le 2C_1\rho_1(c_n a_n^{-1},\infty)\int_E \left(\frac{1}{K}\max_{k=1,\dots,nL}|f_k|\right)^{\alpha-\xi} + \left(\frac{1}{K}\max_{k=1,\dots,nL}|f_k|\right)^{\alpha+\xi} d\mu \\
&\le 2C_1\rho_1(c_n a_n^{-1},\infty)\left(\left(\frac{\|f\|}{K}\right)^{\alpha-\xi} + \left(\frac{\|f\|}{K}\right)^{\alpha+\xi}\right)\mu(\varphi\le nL) \\
&\le C_2\left(\left(\frac{\|f\|}{K}\right)^{\alpha-\xi} + \left(\frac{\|f\|}{K}\right)^{\alpha+\xi}\right),
\end{aligned}
$$

and (3.37) follows.

It remains to consider the processes $\{Z_n^{(1,K)}(t), 0 \le t \le L\}$, $n = 1, 2, \dots$ for a fixed $K > 0$. In the sequel we drop the superscript K for notational convenience. We will show that exist $\gamma_1 > 1$, and $B > 0$ such that for all $0 \le s < t \le L$, $n \ge 1$, $t - s \ge 1/n$ and $\lambda > 0$,

$$
P(|Z_n^{(1)}(t) - Z_n^{(1)}(s)| \ge \lambda) \le \frac{B}{\lambda^2}(t-s)^{\gamma_1}\,. \tag{3.38}
$$

Indeed, by Chebyshev's inequality and the fact that f is supported by A, we see that

$$
\begin{aligned}
P\big(|Z_n^{(1)}(t) - Z_n^{(1)}(s)| \ge \lambda\big) &\le \frac{1}{\lambda^2 c_n^2} E\left|\sum_{k=1}^{\lceil nt\rceil-\lceil ns\rceil}\iint_{|xf_k|\le Kc_n a_n^{-1}} x f_k d\bar N_1\right|^2 \\
&\le \frac{2}{\lambda^2 c_n^2}\sum_{k=1}^{\lceil n(t-s)\rceil}\sum_{l=1}^{\lceil n(t-s)\rceil}\int_E |f_k f_l|\int_0^{Kc_n a_n^{-1}/|f_k|\vee|f_l|} x^2\,\rho_1(dx)\,d\mu\,.
\end{aligned}
$$

It follows from the Potter bounds and the fact that ρ_1 does not assigns mass to the interval $(0,1)$ that for any $0 < \xi < 2-\alpha$ there is $C > 0$ such that for all $a > 0$ large enough and all $r > 0$,

$$
\frac{\int_0^{ra} x^2\,\rho_1(dx)}{\int_0^a x^2\,\rho_1(dx)} \le C\big(r^{2-\alpha-\xi} \vee r^{2-\alpha+\xi}\big)\,.
$$

Therefore, for all n large enough, for some constant $C_1 > 0$,

$$
P\big(|Z_n^{(1)}(t) - Z_n^{(1)}(s)| \ge \lambda\big) \le \frac{C_1}{\lambda^2 c_n^2}\sum_{k=1}^{\lceil n(t-s)\rceil}\sum_{l=1}^{\lceil n(t-s)\rceil}\int_E \frac{|f_k f_l|}{(|f_k|\vee|f_l|)^{2-\alpha-\xi}}\,d\mu\int_0^{c_n a_n^{-1}} x^2\,\rho_1(dx)
$$

$$+\frac{C_1}{\lambda^2 c_n^2}\sum_{k=1}^{\lceil n(t-s)\rceil}\sum_{l=1}^{\lceil n(t-s)\rceil}\int_E \frac{|f_k f_l|}{(|f_k|\vee|f_l|)^{2-\alpha+\xi}}\,d\mu\int_0^{c_n a_n^{-1}} x^2\,\rho_1(dx)\,.$$

Note that by Karamata's theorem, (2.7) and the definition (3.17) of the normalizing sequence (c_n), there is $C_2>0$ such that

$$\int_0^{c_n a_n^{-1}} x^2\,\rho_1(dx)\le C_2\,\frac{c_n^2}{na_n}\,.$$

If $1<\alpha<2$, we impose also the constraint $\xi<\alpha-1$, and use the relation

$$\frac{|f_k f_l|}{(|f_k|\vee|f_l|)^{2-\alpha\pm\xi}}=\big(|f_k|\wedge|f_l|\big)\big(|f_k|\vee|f_l|\big)^{\alpha-1\mp\xi}\,, \tag{3.39}$$

so that

$$\frac{1}{c_n^2}\sum_{k=1}^{\lceil n(t-s)\rceil}\sum_{l=1}^{\lceil n(t-s)\rceil}\int_E \frac{|f_k f_l|}{(|f_k|\vee|f_l|)^{2-\alpha\pm\xi}}\,d\mu\int_0^{c_n a_n^{-1}} x^2\,\rho_1(dx)$$

$$\le C_2\frac{1}{na_n}\sum_{k=1}^{\lceil n(t-s)\rceil}\sum_{l=1}^{\lceil n(t-s)\rceil}\int_E\big(|f_k|\wedge|f_l|\big)\big(|f_k|\vee|f_l|\big)^{\alpha-1\mp\xi}\,d\mu$$

$$\le 2C_2\frac{1}{na_n}\Bigg[\lceil n(t-s)\rceil\int_E|f|^{\alpha\mp\xi}\,d\mu$$

$$+\sum_{k=1}^{\lceil n(t-s)\rceil-1}\sum_{l=k+1}^{\lceil n(t-s)\rceil}\left(\int_E|f_l||f_k|^{\alpha-1\mp\xi}\,d\mu+\int_E|f_k||f_l|^{\alpha-1\mp\xi}\,d\mu\right)\Bigg]$$

$$:=J_n(1)+J_n(2)+J_n(3)\,.$$

The fact that $t-s>1/n$ and (a_n) is regularly varying with the positive exponent β, shows that for any $1<\gamma_1<1+\beta$ there is some constant $C_3>0$, such that for all $n=1,2,\dots$,

$$J_n(1)\le C_3(t-s)^{\gamma_1}\,.$$

Next, by the duality relation (2.1),

$$J_n(2)\le\frac{4C_2}{a_n}(t-s)\sum_{k=1}^{\lceil n(t-s)\rceil}\int_E|f_k||f|^{\alpha-1\mp\xi}\,d\mu$$

$$=\frac{4C_2}{a_n}(t-s)\int_A|f|\left(\sum_{k=1}^{\lceil n(t-s)\rceil}\widehat{T}^k|f|^{\alpha-1\mp\xi}\right)\,d\mu\,.$$

If f is bounded, then by the Darling-Kac property of the set A we have, for some constants $C_4,\ C_5 > 0$,

$$J_n(2) \le C_4(t-s)\frac{a_{\lceil n(t-s)\rceil}}{a_n}\mu(|f|) \le C_5(t-s)^{\gamma_1},\ \ 1<\gamma_1<1+\beta\,,$$

by the regular variation of (a_n). If, on the other hand, A is a uniform set for $|f|$, then we can write

$$\sum_{k=1}^{\lceil n(t-s)\rceil} \widehat{T}^k|f|^{\alpha-1\mp\xi} \le \sum_{k=1}^{\lceil n(t-s)\rceil} \widehat{T}^k\mathbf{1}_A + \sum_{k=1}^{\lceil n(t-s)\rceil} \widehat{T}^k|f|\,,$$

and obtain the same bound on J_2 by using both the Darling-Kac property and the uniform property of the set A. A similar argument shows that, for some constant $C_6 > 0$ we also have

$$J_n(3) \le C_6(t-s)^{\gamma_1},\ \ 1<\gamma_1<1+\beta\,,$$

which proves (3.38) in the case $1<\alpha<2$.

Finally, for $0<\alpha\le 1$ the same argument works, if we replace the relation (3.39) by

$$\frac{|f_kf_l|}{(|f_k|\vee|f_l|)^{1+\xi}} \le \big(|f_k|\wedge|f_l|\big)^{1-\xi},\ \ \frac{|f_kf_l|}{(|f_k|\vee|f_l|)^{1-\xi}} = \big(|f_k|\wedge|f_l|\big)\big(|f_k|\vee|f_l|\big)^{\xi}\,,$$

respectively if $\alpha = 1$, and

$$\frac{|f_kf_l|}{(|f_k|\vee|f_l|)^{2-\alpha\pm\xi}} \le \big(|f_k|\wedge|f_l|\big)^{\alpha\mp\xi}$$

if $0<\alpha<1$. This proves (3.38) in all cases and, hence, completes the proof of the theorem. $\square$

CHAPTER 4

LIMIT THEORY FOR THE SAMPLE COVARIANCE FOR HEAVY TAILED STATIONARY INFINITELY DIVISIBLE PROCESSES GENERATED BY CONSERVATIVE FLOWS

4.1 The setup

We consider an infinitely divisible process

$$X_n = \int_E f \circ T^n(x) dM(x) \,, \quad n = 1, 2, \dots, \tag{4.1}$$

where M is an independently scattered infinitely divisible random measure on a measurable space $(E, \mathcal{E})$ and $f : E \to \mathbb{R}$ is a deterministic function, and $T : E \to E$ is a measurable map. We often denote $f_n(x) = f \circ T^n(x)$. The random measure M is assumed to be homogeneous symmetric and have a local Lévy measure ρ and a σ-finite *infinite* control measure μ. We assume, throughout this chapter, that a Gaussian component is identically zero. By these assumptions on the random measure M, we may write, for every $A \in \mathcal{E}$ of finite μ-measure,

$$Ee^{iuM(A)} = \exp\left\{ -\mu(A) \int_{\mathbb{R}} \big(1 - \cos(ux)\big) \, \rho(dx) \right\} \quad u \in \mathbb{R}.$$

One of the central assumption in our work is related to the heavy tailedness of the process $\mathbf{X} = (X_1, X_2, \dots)$. We assume that ρ has a regularly varying tail with index $-\alpha$, $0 < \alpha < 2$,

$$\rho(\cdot, \infty) \in RV_{-\alpha} \ \text{ at infinity.} \tag{4.2}$$

From now, we express ρ by ρ_α, emphasizing its dependence on the tail parameter α. We will add an extra assumption on the lower tail of the local Lévy measure ρ_α: for some $p_0 \in (0, 2)$,

$$x^{p_0} \rho_\alpha(x, \infty) \to 0 \ \text{ as } x \downarrow 0 \,. \tag{4.3}$$

The assumption that the process $\mathbf{X} = (X_1, X_2, \dots)$ is generated by a conservative flow plays a major role in this chapter as well as in the last one. This assumption is, indeed, related to long memory in the process $\mathbf{X}$; the length of memory observed in the process $\mathbf{X}$ is significantly longer than that in the process generated by a dissipative flow. See for example, Samorodnitsky (2004) and Roy (2008). Now, we let T be a conservative ergodic and measure preserving map on a σ-finite infinite measure space $(E, \mathcal{E}, \mu)$. In Chapter 3, we assumed that T has a Darling-Kac set $A \in \mathcal{E}$, but here, this assumption will be weakened. Namely, we will simply assume that T is pointwise dual ergodic and, hence, T admits some uniform set $A \in \mathcal{E}$ with $0 < \mu(A) < \infty$. We suppose that the normalizing sequence (a_n) for pointwise dual ergodicity is regularly varying with exponent $0 \leq \beta < 1$. Unlike the last chapter, we do not exclude the limiting case $\beta = 0$. In addition, with φ being the first entrance time to A and $A_0 = A$, $A_k = A^c \cap \{\varphi = k\}$, $k \geq 1$, we will generalize (2.29) to the condition that

$$\frac{1}{\mu(\varphi \leq n)} \sum_{k=1}^{n} \widehat{T}^k \mathbf{1}_{A_k}(x) \ \text{ is uniformly bounded on } A\,. \tag{4.4}$$

We will assume that $f : E \to \mathbb{R}$ satisfies

$$f \in L^2(\mu) \ \text{ with } \ \mu(f^2) = \int_E f(x)^2 \mu(dx) > 0 \tag{4.5}$$

and that f is supported by the uniform set A.

Now, due to the assumptions on the local Lévy measure ρ_α and the function f, it is not hard to see that the process $\mathbf{X} = (X_1, X_2, \dots)$ turns out to be a well-defined, symmetric, stationary infinitely divisible process; see Rajput and Rosiński (1989). Moreover, the marginal tails of the process $\mathbf{X}$ have the same asymptotic decaying rate at which the Lévy measure ρ_α decays. The process $\mathbf{X}$, therefore, belongs to the domain of attraction of a SαS law. See Rosiński and Samorodnitsky (1993) and Section 3.1 of this dissertation for more details.

As in Chapter 3, the assumptions on the flow (T^n) and regular variation of the local Lévy measure ρ_α mostly characterize the type of limit theorems of the sample autocovariances. As in Chapter 3, once again, the marginal tails of the process $\mathbf{X}$ and its length of memory are parameterized by the two exponents $\alpha \in (0,2)$ and $\beta \in [0,1)$, respectively.

4.2 Limit Theorem on the Sample Autocovariances

This section presents the main limit theorems on the sample autocovariances and the sample autocorrelations of the process $\mathbf{X} = (X_1, X_2, \dots)$ given in (4.1). We, first, want to define some normalizing sequence (c_n), which can capture how rapidly the sample autocovariances of the process $\mathbf{X}$ grow.

Let $U_\alpha(x) = \rho_\alpha(x,\infty)$, $x > 0$. We define the right continuous inverse of $U_\alpha(x)$ by

$$U_\alpha^{\leftarrow}(y) = \inf\{x > 0 : U_\alpha(x) \le y\}\,, \quad y > 0\,.$$

Given the normalizing sequence (a_n) for pointwise dual ergodicity and its wandering rate sequence (w_n), we define

$$c_n = 2^{2/\alpha} C_{\alpha,\beta} C_{\alpha/2}^{-2/\alpha} a_n \big(U_\alpha^{\leftarrow}(w_n^{-1})\big)^2\,, \tag{4.6}$$

where

$$C_{\alpha,\beta} = \Gamma(1+\beta)\big(EM_\beta(1-V_\beta)^{\alpha/2}\big)^{2/\alpha}\,.$$

Here $M_\beta(t)$ is the Mittag-Leffler process defined in (2.10), and V_β is a random variable defined in (2.34) and is independent of $M_\beta(t)$. Note that if $\beta = 0$, $M_\beta(1-V_\beta)$ can be interpreted as the standard exponential random variable. On the other hand, $C_{\alpha/2}$ is a tail constant for an $\alpha/2$-stable random variable (see (3.18)). It then follows from Proposition 4.2.2 below that (c_n) satisfies the asymptotic relation

$$\rho_\alpha\big((c_n a_n^{-1})^{1/2},\infty\big) \sim 2^{-1} C_{\alpha/2}\big(\mu(f^2) a_n\big)^{\alpha/2} \Big(\int_E \Big|\sum_{k=1}^n f_k(x)^2\Big|^{\alpha/2} \mu(dx)\Big)^{-1} \quad \text{as } n \to \infty\,. \tag{4.7}$$

From the definition (4.6), it is easy to obtain the regular variation exponent for (c_n):

$$c_n \in RV_{\beta+2(1-\beta)/\alpha}\,. \tag{4.8}$$

Therefore, the growth rate of the sample autocovariance of the process $\mathbf{X}$ is determined by not only heavy tailedness of the marginals but also the length of memory. This is in contrast to the case of the processes generated by dissipative flows, e.g., α-stable moving averages studied by Resnick et al. (1999), where it was shown that the sample autocovariances of the α-stable moving averages grow at a regularly varying rate with exponent $2/\alpha$. A substitution of $\beta = 0$ into (4.8) yields $c_n \in RV_{2/\alpha}$, which implies that $\beta = 0$ corresponds to the shortest memory in the process $\mathbf{X}$. As β gets closer to 1, it is expected to exhibit longer memory.

Our argument for the main limit theorems on the sample autocovariances will be separated into two cases. First, we discuss the case where α and β lie in the range

$$\text{either } 1 < \alpha < 2,\ 0 \leq \beta < 1 \ \text{ or } \ 0 < \alpha \leq 1,\ 0 \leq \beta < 1/(2-\alpha)\,. \tag{4.9}$$

In this case, the main theorem will be proved by a series of propositions (Propositions 4.2.3 - 4.2.7), together with Lemma 4.4.1. If α and β lie outside the range (4.9), in addition to the abovementioned propositions, we will apply Lemma 4.4.2 or 4.4.3.

Finally, we denote the sample autocovariance of the process $\mathbf{X}$ by

$$\widehat{\gamma}_n(h) = \frac{1}{n}\sum_{k=1}^{n} X_k X_{k+h}, \quad h = 0, 1, 2, \ldots,$$

and the sample autocorrelation function by $\widehat{\rho}_n(h) = \widehat{\gamma}_n(h)/\widehat{\gamma}_n(0)$, $h = 0, 1, 2, \ldots$.

Theorem 4.2.1. *Let T be a conservative ergodic and measure preserving map on a σ-finite infinite measure space $(E, \mathcal{E}, \mu)$. We assume that T is a pointwise dual ergodic map with normalizing sequence $(a_n) \in RV_\beta$. Suppose that T admits a uniform set $A \in \mathcal{E}$, $0 < \mu(A) < \infty$, and that (4.4) is fulfilled.*
Let M be a symmetric homogeneous infinitely divisible random measure on $(E, \mathcal{E})$ with control measure μ and local Lévy measure ρ_α, which satisfies (4.2) and (4.3).
Let $f : E \to \mathbb{R}$ be a measurable function that is supported by the uniform set A and satisfies integrability condition (4.5).

Let α and β lie in the range (4.9). Then the stationary infinitely divisible process $\mathbf{X}$ given in (4.1) satisfies for $H \geq 0$,

$$\left(\frac{n}{c_n}\widehat{\gamma}_n(h),\, h = 0, \ldots, H\right) \Rightarrow \big(\mu(f \cdot f_h)W,\, h = 0, \ldots, H\big)\,. \tag{4.10}$$

Here, W is a positive strictly stable random variable of exponent $\alpha/2$, i.e., the characteristic function of W is given by

$$Ee^{iuW} = \exp\{\int_{(0,\infty)} (e^{iux} - 1)\rho_*(dx)\} \quad u \in \mathbb{R}\,, \tag{4.11}$$

with $\rho_*(dx) = 2^{-1}\alpha C_{\alpha/2} x^{-1-\alpha/2} dx$, $x > 0$.

As a consequence, we also get

$$\widehat{\rho}_n(h) \xrightarrow{p} \frac{\mu(f \cdot f_h)}{\mu(f^2)}\,. \tag{4.12}$$

On the other hand, if α and β lie outside the range (4.9), we additionally suppose either (i) or (ii) below.

(i): $T \times T$ is still a conservative and ergodic map on $(E \times E, \mathcal{E} \times \mathcal{E}, \mu \times \mu)$. Moreover, $T \times T$ is a pointwise dual ergodic map with normalizing sequence $(a'_n) \in RV_{2\beta-1}$, and further, we extend the condition (4.4) to a two-dimensional version:

$$\frac{1}{(\mu \times \mu)(\varphi(x,y) \leq n)} \sum_{k=1}^{n} (\widehat{T \times T})^k \mathbf{1}_{(A \times A)^c \cap \{\varphi(x,y) = k\}} \quad \text{is uniformly bounded on } A \times A\,, \tag{4.13}$$

where $\varphi(x,y) = \min\{n \geq 1 : (T^n x, T^n y) \in A \times A\}$ is the first entrance time to the set $A \times A$, and $\widehat{T \times T}$ is a dual operator for $T \times T$.

(ii): A is a uniformly returning set for $\mathbf{1}_A$, i.e., there exists an increasing normalizing sequence (b_n) such that

$$b_n \widehat{T}^n \mathbf{1}_A \to \mu(A) \quad \text{uniformly, a.e. on } A\,. \tag{4.14}$$

Moreover, f is a bounded function.

Then, (4.10) and (4.12) follow again.

We will first start by checking how rapidly the sample autocovariance of the process $\mathbf{X}$ grows when it is associated with a conservative flow.

Proposition 4.2.2. *Under the assumptions of Theorem 4.2.1,*

$$\left(\int_E |S_n(f^2)|^{\alpha/2} d\mu\right)^{2/\alpha} \sim \mu(f^2) C_{\alpha,\beta} a_n w_n^{2/\alpha},$$

where

$$C_{\alpha,\beta} = \Gamma(1+\beta)\big(EM_\beta(1-V_\beta)^{\alpha/2}\big)^{2/\alpha}. \tag{4.15}$$

Proof. We write

$$\left(\int_E |S_n(f^2)|^{\alpha/2} d\mu\right)^{2/\alpha} = a_n \mu(\varphi \le n)^{2/\alpha} \left(\int_E \left|\frac{S_n(f^2)}{a_n}\right|^{\alpha/2} d\mu_n\right)^{2/\alpha},$$

where $\mu(\cdot) = \mu\big(\cdot \cap \{\varphi \le n\}\big)/\mu(\varphi \le n)$. Because of (2.7) and (2.8),

$$\sup_{n\ge 1} \int_E \left|\frac{S_n(f^2)}{a_n}\right| d\mu_n \le \sup_{n \ge 1} \frac{n}{a_n \mu(\varphi \le n)} \int_E f^2 d\mu < \infty.$$

Hence, $(|S_n(f^2)/a_n|^{\alpha/2},\ n \ge 1)$ is a uniformly integrable sequence with respect to μ_n. Therefore, Proposition 2.2.5 implies

$$\left(\int_E \left|\frac{S_n(f^2)}{a_n}\right|^{\alpha/2} d\mu_n\right)^{2/\alpha} \to \mu(f^2) C_{\alpha,\beta}.$$

□

As a preparation for the proof of Theorem 4.2.1, we decompose the process $\mathbf{X}$ by the magnitude of the Lévy jumps: we first let

$$\rho_{\alpha,1}(\cdot) = \rho_\alpha(\cdot \cap \{x : |x| > 1\}),$$

$$\rho_{\alpha,2}(\cdot) = \rho_\alpha(\cdot \cap \{x : |x| \le 1\}).$$

Let M_i, $i = 1, 2$, denote homogeneous symmetric infinitely divisible random measures with the same control measure μ and local Lévy measures $\rho_{\alpha,i}$, $i = 1, 2$. Then, X_n can be written as

$$X_n \stackrel{d}{=} \int_E f_n(x) dM_1(x) + \int_E f_n(x) dM_2(x).$$

Denote $X_n^{(i)} = \int_E f_n(x) dM_i(x)$, $i = 1, 2$. We may write

$$n\widehat{\gamma}_n(h) = \sum_{k=1}^n X_k X_{k+h} \stackrel{d}{=} \sum_{k=1}^n X_k^{(1)} X_{k+h}^{(1)} + \sum_{k=1}^n X_k^{(1)} X_{k+h}^{(2)} + \sum_{k=1}^n X_k^{(2)} X_{k+h}^{(1)} + \sum_{k=1}^n X_k^{(2)} X_{k+h}^{(2)} .$$

The main tool used in our proof is a certain series representation of (X_n), which was developed by Rosiński (1990). We also refer to Section 3.10 in Samorodnitsky and Taqqu (1994). Since μ is a σ-finite measure, one can find an μ-equivalent probability measure μ_0 such that

$$\mu_0(B) = \int_B q(x)\mu(dx) ,$$

where q is a positive measurable function on E. For $l = 1, 2$, we write $U_{\alpha,l}(x) = \rho_{\alpha,l}(x, \infty)$ for $x > 0$ and define the right continuous inverse of $U_{\alpha,l}(x)$ by

$$U_{\alpha,l}^{\leftarrow}(y) = \inf\{x > 0 : U_{\alpha,l}(x) \le y\}, \quad y > 0 .$$

According to Rosiński (1990), $X_n^{(l)}$ can be represented in law as

$$(X_n^{(l)},\, n \ge 0) \stackrel{d}{=} \left(\sum_{i=1}^\infty \epsilon_i U_{\alpha,l}^{\leftarrow}\left(\frac{\Gamma_i q(V_i)}{2}\right) f_n(V_i),\, n \ge 0 \right) ,$$

where (ϵ_i) is an i.i.d. Rademacher sequence taking 1 or -1 with probability $1/2$, Γ_i is the ith jump time of a unit rate Poisson process, and (V_i) is a sequence of i.i.d. random variables with common distribution μ_0.

In the next proposition, the series representation for $\sum_{k=1}^n X_k^{(l)} X_{k+h}^{(l)}$ will be split into a diagonal part and an off-diagonal part, and the diagonal part can be represented as a specific stochastic integral driven by a positive infinitely divisible random measure.

Proposition 4.2.3. *For any $H \ge 0$, $n > 0$, and $l = 1, 2$,*

$$\left(\sum_{k=1}^n X_k^{(l)} X_{k+h}^{(l)},\, h = 0, \ldots, H \right) \stackrel{d}{=} \left(Y'_{n,l}(h) + Y''_{n,l}(h),\, h = 0, \ldots, H \right)$$

with

$$Y'_{n,l}(h) = \int_E \sum_{k=1}^n f_k(x) f_{k+h}(x) d\widetilde{M}_l(x) ,$$

$$Y''_{n,l}(h) = \sum_{i \ne j} \epsilon_i \epsilon_j U_{\alpha,l}^{\leftarrow}\left(\frac{\Gamma_i q(V_i)}{2}\right) U_{\alpha,l}^{\leftarrow}\left(\frac{\Gamma_j q(V_j)}{2}\right) \sum_{k=1}^n f_k(V_i) f_{k+h}(V_j) .$$

Here, $\widetilde{M}_l$ is a positive infinitely divisible random measure defined by

$$Ee^{iu\widetilde{M}_l(A)} = \exp\{\mu(A) \int_{(0,\infty)} (e^{iux} - 1)\widetilde{\rho}_{\frac{\alpha}{2},l}(dx)\}\,, \quad u \in \mathbb{R}\,,$$

where $\widetilde{\rho}_{\frac{\alpha}{2},l}$ is a local Lévy measure concentrated on the positive half-line such that

$$\widetilde{\rho}_{\frac{\alpha}{2},l}(x,\infty) = 2\rho_{\alpha,l}(x^{1/2},\infty) \quad \text{for } x > 0\,. \tag{4.16}$$

Proof. Since

$$\sum_{k=1}^{n} X_k^{(l)} X_{k+h}^{(l)} \stackrel{d}{=} \sum_{i=1}^{\infty} U_{\alpha,l}^{\leftarrow}\left(\frac{\Gamma_i q(V_i)}{2}\right)^2 \sum_{k=1}^{n} f_k(V_i) f_{k+h}(V_i) + Y_{n,l}''(h)\,,$$

it suffices to show that the first term converges a.s. and is distributionally equal to $Y_{n,l}'(h)$. For this purpose, from Rosiński (1990), we only check that

$$\int_0^\infty P\left(U_{\alpha,l}^{\leftarrow}\left(\frac{rq(V_1)}{2}\right)^2 \sum_{k=1}^{n} f_k(V_1) f_{k+h}(V_1) \in \cdot\right) dr \tag{4.17}$$

$$= (\widetilde{\rho}_{\frac{\alpha}{2},l} \times \mu)\{(v,x) : v \sum_{k=1}^{n} f_k(x) f_{k+h}(x) \in \cdot\}$$

and

$$\int_E \int_{\mathbb{R}} \min\left(1, \left|v \sum_{k=1}^{n} f_k(x) f_{k+h}(x)\right|\right) \widetilde{\rho}_{\frac{\alpha}{2},l}(dv)\mu(dx) < \infty\,. \tag{4.18}$$

Note that the right hand side of (4.17) is exactly the Lévy measure of $Y_{n,l}'(h)$.

Since a simple calculation verifies (4.17), we only prove (4.18). By regular variation of the local Lévy measure ρ_α, the Potter bound (see e.g., Proposition 0.8 in Resnick (1987)) provides

$$\widetilde{\rho}_{\frac{\alpha}{2},1}(x,\infty) \le C_1 x^{-(\alpha-\xi)/2}, \quad x > 0$$

for some constants $0 < \xi < \alpha$ and $C_1 > 0$. Also by (4.3), we get an obvious upper bound; for some $C_2 > 0$,

$$\widetilde{\rho}_{\frac{\alpha}{2},2}(x,\infty) \le C_2 x^{-p_0/2}, \quad x > 0\,.$$

These bounds, together with the fact that f has a support of finite μ-measure and $f \in L^2(\mu)$, can establish (4.18). $\square$

Now the following expression has been justified:

$$\frac{n}{c_n}\widehat{\gamma}_n(h) \stackrel{d}{=} c_n^{-1}\Big(Y'_{n,1}(h) + Y''_{n,1}(h) + \sum_{k=1}^{n} X_k^{(1)}X_{k+h}^{(2)} + \sum_{k=1}^{n} X_k^{(2)}X_{k+h}^{(1)} + \sum_{k=1}^{n} X_k^{(2)}X_{k+h}^{(2)}\Big)\,.$$

We will describe the idea of the proof of Theorem 4.2.1. First, we will verify by Proposition 4.2.4 below that

$$\left(\frac{Y'_{n,1}(h)}{c_n},\, h = 0,\dots,H\right) \Rightarrow \big(\mu(f\cdot f_h)W,\, h=0,\dots,H\big)\,,$$

where W is defined in (4.11). Subsequently, Proposition 4.2.5 will prove $Y''_{n,1}(h)/c_n \xrightarrow{p} 0$ for every $h \geq 0$. On the other hand, applications of the Cauchy-Schwarz inequality and the result of Proposition 4.2.3 yield

$$\begin{aligned}\Big|\sum_{k=1}^{n} X_k^{(1)}X_{k+h}^{(2)}\Big| &\leq \left(\sum_{k=1}^{n}(X_k^{(1)})^2\right)^{1/2}\left(\sum_{k=1}^{n}(X_{k+h}^{(2)})^2\right)^{1/2}\\ &\stackrel{d}{=} \left(\sum_{k=1}^{n}(X_k^{(1)})^2\right)^{1/2}(Y'_{n,2}(0)+Y''_{n,2}(0))^{1/2}.\end{aligned}$$

Thus, to complete the proof of Theorem 4.2.1, we need to show $(Y'_{n,2}(0)+Y''_{n,2}(0))/c_n \xrightarrow{p} 0$, which will be established by Propositions 4.2.6 and 4.2.7 below. It is noteworthy that Proposition 2.2.5 is needed for the proof of Proposition 4.2.4, while Propositions 4.2.5 and 4.2.7 use the results of Lemmas 4.4.1 - 4.4.4 in the Appendix of this chapter.

Proposition 4.2.4. *For any $H \geq 0$,*

$$\left(\frac{Y'_{n,1}(h)}{c_n},\, h = 0,\dots,H\right) \Rightarrow \big(\mu(f\cdot f_h)W,\, h=0,\dots,H\big)\,,$$

where W is defined in (4.11).

Proof. By virtue of the Cramer-Wold device, we only have to show

$$\frac{1}{c_n}\sum_{h=0}^{H}\theta_h Y'_{n,1}(h) \Rightarrow \sum_{h=0}^{H}\theta_h\mu(f\cdot f_h)W \;\text{ in } \mathbb{R} \tag{4.19}$$

for every $\theta_0,\dots,\theta_H \in \mathbb{R}$. Let $\phi(x) = f(x)\sum_{h=0}^{H}\theta_h f_h(x)$, and denote $S_n(\phi)(x) = \sum_{k=1}^{n}\phi\circ T^k(x)$. Then, (4.19) is equivalent to

$$\frac{1}{c_n}\int_E S_n(\phi)(x)d\widetilde{M}_1(x) \Rightarrow \mu(\phi)W \quad \text{in } \mathbb{R}\,. \tag{4.20}$$

A sufficient condition for weak convergence of the left hand side in (4.20) reduces to the following (see e.g., Theorem 13.14 in Kallenberg (1997)): for every $r > 0$,

$$\int_E \left(\frac{S_n(\phi)}{c_n}\right)^2 \int_0^{rc_n|S_n(\phi)|^{-1}} x\widetilde{\rho}_{\frac{\alpha}{2},1}(x,\infty)dxd\mu \to \frac{r^{2-\alpha/2}C_{\alpha/2}}{2-\alpha/2}|\mu(\phi)|^{\alpha/2}\,, \tag{4.21}$$

$$\int_E \widetilde{\rho}_{\frac{\alpha}{2},1}(rc_n|S_n(\phi)|^{-1},\infty)d\mu \to r^{-\alpha/2}C_{\alpha/2}|\mu(\phi)|^{\alpha/2}\,, \tag{4.22}$$

and

$$\int_E \frac{S_n(\phi)}{c_n} \int_0^{c_n|S_n(\phi)|^{-1}} \widetilde{\rho}_{\frac{\alpha}{2},1}(x,\infty)dxd\mu \to \frac{2C_{\alpha/2}}{2-\alpha} sgn(\mu(\phi))|\mu(\phi)|^{\alpha/2} \tag{4.23}$$

($sgn(u) = u/|u|$ if $u \neq 0$ and $sgn(0) = 0$). We only prove (4.21), because (4.22) and (4.23) can be handled analogously.

For (4.21), we need to use the result in Proposition 2.2.5

$$\frac{S_n(\phi)}{a_n} \Rightarrow \mu(\phi)\Gamma(1+\beta)M_\beta(1-V_\beta) \text{ in } \mathbb{R}\,,$$

where the weak convergence takes place under a probability measure $\mu_n(\cdot) = \mu(\cdot \cap \{\varphi \le n\})/\mu(\varphi \le n)$. $(M_\beta(t))$ is the Mittag-Leffler process defined on some probability space $(\Omega', \mathcal{F}', P')$, and V_β is a random variable defined on the same probability space, independent of $M_\beta(t)$, such that $P'(V_\beta \le x) = x^{1-\beta}$, $0 \le x \le 1$.

Applying the Skorohod's embedding theorem, there exist random variables Y and Y_n, $n = 1, 2, \ldots$ defined on some probability space $(\Omega^*, \mathcal{F}^*, P^*)$ such that

$$P^* \circ Y_n^{-1} = \mu_n \circ \left(\frac{S_n(\phi)}{a_n}\right)^{-1}, \quad n = 1, 2, \ldots,$$

$$P^* \circ Y^{-1} = P' \circ \left(\mu(\phi)\Gamma(1+\beta)M_\beta(1-V_\beta)\right)^{-1},$$

$$Y_n \to Y \quad P^*\text{-a.s.}.$$

Let $\psi(y) = y^{-2}\int_0^{ry} x\widetilde{\rho}_{\frac{\alpha}{2},1}(x,\infty)dx$, then we can proceed

$$\int_E \left(\frac{S_n(\phi)}{c_n}\right)^2 \int_0^{rc_n|S_n(\phi)|^{-1}} x\widetilde{\rho}_{\frac{\alpha}{2},1}(x,\infty)dxd\mu = \int_E \psi\left(\frac{c_n}{|S_n(\phi)|}\right)d\mu$$

$$= \mu(\varphi \le n)E^*\left[\psi\left(\frac{c_n}{a_n|Y_n|}\right)\right].$$

It follows from (4.8) that $c_n a_n^{-1}|Y_n|^{-1} \to \infty$, P^*-a.s.. Therefore, Karamata's theorem (see e.g., Theorem 0.6 in Resnick (1987)) yields

$$\psi\left(\frac{c_n}{a_n|Y_n|}\right) \sim \frac{r^2}{2-\alpha/2}\widetilde{\rho}_{\frac{\alpha}{2},1}(rc_n a_n^{-1}|Y_n|^{-1},\infty) \quad \text{as } n\to\infty,\ P^*\text{-a.s.}$$

From uniform convergence theorem of regularly varying functions of negative indices (see e.g., Proposition 0.5 in Resnick (1987)), we can say that

$$\widetilde{\rho}_{\frac{\alpha}{2},1}(rc_n a_n^{-1}|Y_n|^{-1},\infty) \sim r^{-\alpha/2}|Y_n|^{\alpha/2}\widetilde{\rho}_{\frac{\alpha}{2},1}(c_n a_n^{-1},\infty) \quad \text{as } n\to\infty, \quad P^*\text{-a.s.}$$

From (4.7), (4.16), and Proposition 4.2.2,

$$\begin{aligned}\mu(\varphi\le n)\psi\left(\frac{c_n}{a_n|Y_n|}\right) &\sim \frac{r^{2-\alpha/2}}{2-\alpha/2}\mu(\varphi\le n)|Y_n|^{\alpha/2}\widetilde{\rho}_{\frac{\alpha}{2},1}(c_n a_n^{-1},\infty)\\ &\to \frac{r^{2-\alpha/2}C_{\alpha/2}}{2-\alpha/2}C_{\alpha,\beta}^{-\alpha/2}|Y|^{\alpha/2} \quad \text{as } n\to\infty, \quad P^*\text{-a.s.}\end{aligned}$$

Integrating the limit yields

$$E^*\left[\frac{r^{2-\alpha/2}C_{\alpha/2}}{2-\alpha/2}C_{\alpha,\beta}^{-\alpha/2}|Y|^{\alpha/2}\right] = \frac{r^{2-\alpha/2}C_{\alpha/2}}{2-\alpha/2}|\mu(\phi)|^{\alpha/2},$$

which is exactly the right hand side of (4.21). Now, to complete the proof, we need to justify taking the limit under the integral. For this, we will apply Pratt's lemma (see Pratt (1960)). According to Pratt's lemma, we must find a sequence of measurable functions $G_0, G_1, \ldots$ defined on $(\Omega^*, \mathcal{F}^*, P^*)$ such that

$$\mu(\varphi\le n)\psi\left(\frac{c_n}{a_n|Y_n|}\right) \le G_n \quad P^*\text{-a.s.}, \quad n=1,2,\ldots, \tag{4.24}$$

$$G_n \to G_0 \quad \text{as } n\to\infty \quad P^*\text{-a.s., and} \tag{4.25}$$

$$E^*G_n \to E^*G_0 \quad \text{as } n\to\infty. \tag{4.26}$$

For (4.24), there is a $C_1>0$ such that

$$\mu(\varphi\le n)\psi\left(\frac{c_n}{a_n|Y_n|}\right) \le C_1\frac{\psi(c_n a_n^{-1}|Y_n|^{-1})}{\psi(c_n a_n^{-1})}$$

because $\mu(\varphi\le n)\psi(c_n a_n^{-1})$ has a positive and finite limit.

Applying the Potter bound, for any fixed $0<\xi<\min(\alpha,2-\alpha)$, we have

$$\frac{\psi(c_n a_n^{-1}|Y_n|^{-1})}{\psi(c_n a_n^{-1})}\mathbf{1}_{\{c_n>a_n|Y_n|\}} \le C_2(|Y_n|^{(\alpha-\xi)/2}+|Y_n|^{(\alpha+\xi)/2})$$

for some $C_2 > 0$.

Since ψ is bounded on $(0,1]$, for some constant $C_3 \geq C_2$,

$$\frac{\psi(c_n a_n^{-1}|Y_n|^{-1})}{\psi(c_n a_n^{-1})}\mathbf{1}_{\{c_n \leq a_n|Y_n|\}} \leq \frac{C_3}{\psi(c_n a_n^{-1})}\frac{a_n}{c_n}|Y_n|\,.$$

Therefore, we may write

$$\mu(\varphi \leq n)\psi\left(\frac{c_n}{a_n|Y_n|}\right) \leq C_3\left(|Y_n|^{(\alpha-\xi)/2} + |Y_n|^{(\alpha+\xi)/2} + \frac{a_n}{c_n}\frac{|Y_n|}{\psi(c_n a_n^{-1})}\right)\,.$$

Now, (4.24) is obtained by taking

$$G_n = C_3\left(|Y_n|^{(\alpha-\xi)/2} + |Y_n|^{(\alpha+\xi)/2} + \frac{a_n}{c_n}\frac{|Y_n|}{\psi(c_n a_n^{-1})}\right), \quad n = 1, 2, \ldots .$$

Let

$$G_0 = C_3\left(|Y|^{(\alpha-\xi)/2} + |Y|^{(\alpha+\xi)/2}\right)\,.$$

We know that $a_n c_n^{-1} \in RV_{-2(1-\beta)/\alpha}$ and $\psi(c_n a_n^{-1}) \in RV_{\beta-1}$; thus,

$$\frac{a_n}{c_n}\frac{1}{\psi(c_n a_n^{-1})} \to 0$$

from which (4.25) follows.

To show (4.26), recall that $\sup_{n\geq 1} E^*|Y_n| < \infty$ (see the proof of Proposition 4.2.2). Thus, $(|Y_n|^{(\alpha\pm\xi)/2},\, n \geq 1)$ is uniformly integrable with respect to P^*, which in turn implies (4.26). Now, Pratt's lemma is applicable and (4.21) is complete. □

Proposition 4.2.5.

$$\frac{1}{c_n}Y''_{n,1}(h) \xrightarrow{p} 0\,, \quad h = 0, 1, 2 \ldots .$$

Proof. Choose $\xi > 0$ as specified in Lemma 4.4.1, 4.4.2, or 4.4.3 in accordance with the values of α and β. Let $\alpha' = \alpha - \xi$. For $i \neq j$, we set

$$W_{ij}^{(n,\alpha')} = \frac{1}{c_n}\sum_{k=1}^{n} f_k(V_i) f_{k+h}(V_j) q(V_i)^{-1/\alpha'} q(V_j)^{-1/\alpha'}\,.$$

Because of those lemmas, we eventually obtain

$$E\left|W_{ij}^{(n,\alpha')}\right|^{\alpha'} \to 0\,, \quad \text{for } i \neq j\,. \tag{4.27}$$

We will basically follow the argument in Proposition 4.3 of Resnick et al. (1999). Recall that $Y''_{n,1}(h)/c_n$ can be represented by doubly infinite series

$$\frac{1}{c_n}Y''_{n,1}(h) = \sum_{i\neq j} \epsilon_i\epsilon_j U^{\leftarrow}_{\alpha,1}\left(\frac{\Gamma_i q(V_i)}{2}\right) U^{\leftarrow}_{\alpha,1}\left(\frac{\Gamma_j q(V_j)}{2}\right) \frac{1}{c_n}\sum_{k=1}^{n} f_k(V_i)f_{k+h}(V_j) \equiv \sum_{i\neq j} \widetilde{W}^{(n)}_{ij}\,.$$

Owing to symmetry of the doubly infinite sum, we only have to consider the case $i<j$, and we will indeed show that $\sum_{i<j}\widetilde{W}^{(n)}_{ij} \xrightarrow{p} 0$. According to Lemma 4.4.4, there exist an integer m_0 and constants $C>0$ and $\gamma<\alpha'$ such that for any $m\geq m_0$, all the inequalities given in (a) and (b) of Lemma 4.4.4 hold.

We then decompose $\sum_{i<j}\widetilde{W}^{(n)}_{ij}$ into three summands

$$\sum_{i<j}\widetilde{W}^{(n)}_{ij} = \sum_{i=1}^{m_0}\sum_{j=i+1}^{m_0}\widetilde{W}^{(n)}_{ij} + \sum_{i=1}^{m_0}\sum_{j=m_0+1}^{\infty}\widetilde{W}^{(n)}_{ij} + \sum_{m_0<i<j<\infty}\widetilde{W}^{(n)}_{ij}\,.$$

Now, we only need to prove the following:

$$(i):\ \widetilde{W}^{(n)}_{ij} \xrightarrow{p} 0 \ \text{ for all } i,j;$$

$$(ii):\ \sum_{j=m_0+1}^{\infty}\widetilde{W}^{(n)}_{ij} \xrightarrow{p} 0 \ \text{ for all } i;$$

$$(iii):\ \sum_{m_0<i<j<\infty}\widetilde{W}^{(n)}_{ij} \xrightarrow{p} 0\,.$$

By the bound $U^{\leftarrow}_{\alpha,1}(x) < Cx^{-1/\alpha'}$ and (4.27), it is evident that $\widetilde{W}^{(n)}_{ij}$ converges to 0 in probability, which proves (i). For (ii) and (iii), by virtue of the inequalities given in Lemma 4.4.4, it suffices to show that

$$E(|W^{(n,\alpha')}_{ij}|^{\alpha'}(1+\ln^2_+|W^{(n,\alpha')}_{ij}|)) \to 0\,,\quad i\neq j\,.$$

To show this, let

$$B^{(n,\alpha')}(x,y) = \frac{1}{c_n}\sum_{k=1}^{n} f_k(x)f_{k+h}(y)q(x)^{-1/\alpha'}q(y)^{-1/\alpha'}.$$

Then, we can show that

$$\sup_{x,y\in E}|B^{(n,\alpha')}(x,y)| = \mathcal{O}(n^{2(1-\beta)(1/\alpha'-1/\alpha)}).$$

For the proof, it is important to note that the choice of the density q does not affect the distribution of $Y''_{n,1}(h)$; therefore, we can particularly take

$$q(x) = Q(x)\left(\int_E Q(u)d\mu\right)^{-1},$$

where

$$Q(x) = \max\Big(q_0(x), \big(\sum_{k=1}^{n+h} f_k(x)^2\big)^{\alpha'/2}\Big).$$

Here, $q_0 : E \to (0,\infty)$ is an arbitrarily selected, strictly positive density.

By the Cauchy-Schwarz inequality,

$$\sup_{x,y\in E}|B^{(n,\alpha')}(x,y)| \le \sup_{x,y\in E}\frac{1}{c_n}\big(\sum_{k=1}^{n} f_k(x)^2\big)^{1/2}\big(\sum_{k=1}^{n} f_{k+h}(y)^2\big)^{1/2} q(x)^{-1/\alpha'}q(y)^{-1/\alpha'}$$

$$\le \frac{1}{c_n}\Big(1+\int_E\big(\sum_{k=1}^{n+h} f_k(x)^2\big)^{\alpha'/2}d\mu\Big)^{2/\alpha'} \in RV_{2(1-\beta)(\frac{1}{\alpha'}-\frac{1}{\alpha})},$$

where the last regular variation index is obtained from (4.8).

Now, we have

$$E\big(|W_{ij}^{(n,\alpha')}|^{\alpha'}(1+\ln_+^2|W_{ij}^{(n,\alpha')}|)\big) \le \big(1+\ln_+^2\sup_{x,y\in E}|B^{(n,\alpha')}(x,y)|\big)E|W_{ij}^{(n,\alpha')}|^{\alpha'}.$$

Observing that $E|W_{ij}^{(n,\alpha')}|^{\alpha'}$ has a negative regular variation index (see the proofs of Lemmas 4.4.1, 4.4.2, and 4.4.3), the right hand side vanishes as $n \to \infty$. $\square$

The combination of the next two propositions will prove $(Y'_{n,2}(0)+Y''_{n,2}(0))/c_n \xrightarrow{p} 0$, which can finish the proof of Theorem 4.2.1.

Proposition 4.2.6.

$$\frac{1}{c_n}Y'_{n,2}(0) = \frac{1}{c_n}\int_E S_n(f^2)(x)d\widetilde{M}_2(x) \xrightarrow{p} 0.$$

Proof. From the standard argument for convergence in law of the sequence of infinitely divisible random variables (see e.g., Theorem 13.14 in Kallenberg (1997)), we only have to check

$$\int_E\left(\frac{S_n(f^2)}{c_n}\right)^2\int_0^{c_nS_n(f^2)^{-1}} x\widetilde{\rho}_{\frac{\alpha}{2},2}(x,\infty)dxd\mu \to 0,$$

$$\int_E \widetilde{\rho}_{\frac{\alpha}{2},2}(c_n S_n(f^2)^{-1}, \infty) d\mu \to 0\,,$$

and

$$\int_E \frac{S_n(f^2)}{c_n} \int_0^{c_n S_n(f^2)^{-1}} \widetilde{\rho}_{\frac{\alpha}{2},2}(x, \infty) dx d\mu \to 0\,.$$

However, an obvious upper bound $\widetilde{\rho}_{\frac{\alpha}{2},2}(x,\infty) \le C x^{-p_0/2}$, $x > 0$, and the integrability condition $f \in L^2(\mu)$ easily prove these limits. □

Proposition 4.2.7.

$$\frac{1}{c_n} Y''_{n,2}(0) = \sum_{i \neq j} \epsilon_i \epsilon_j U^{\leftarrow}_{\alpha,2}\left(\frac{\Gamma_i q(V_i)}{2}\right) U^{\leftarrow}_{\alpha,2}\left(\frac{\Gamma_j q(V_j)}{2}\right) \frac{1}{c_n} \sum_{k=1}^{n} f_k(V_i) f_k(V_j) \xrightarrow{p} 0\,.$$

Proof. The proof is analogous to that of Proposition 4.2.5. Taking advantage of the inequalities given in Lemma 4.4.4 (see also Remark 4.4.5), the proof will be finished if

$$E\big(|W_{ij}^{(n,p_0)}|^{p_0}(1 + \ln_+^2 |W_{ij}^{(n,p_0)}|)\big) \to 0\,, \quad i \neq j\,.$$

The argument for showing this is mostly the same as in Proposition 4.2.5, and so we omit it. □

4.3 Examples

We present three examples of different situations where Theorem 4.2.1 applies. The first example is what Resnick et al. (2000) studied (see also Example 3.3.5), but it can be regarded as a special case of our more general setup.

Example 4.3.1. Let $(x_k, k \ge 0)$ be an irreducible null recurrent Markov chain with state space $\mathbb{Z}$ and transition matrix $P = (p_{ij})$. Let $P_i(\cdot)$ be a probability law of (x_k) starting in state $i \in \mathbb{Z}$. Since (x_k) is null recurrent, there exists a unique (up to constant multiplications), infinite, invariant measure (π_i). We set $\pi_0 = 1$ for normalization. Define a σ-finite and infinite measure on $(E, \mathcal{E}) = (\mathbb{Z}^{\mathbb{N}}, \mathcal{B}(\mathbb{Z}^{\mathbb{N}}))$ by

$$\mu(\cdot) = \sum_{i \in \mathbb{Z}} \pi_i P_i(\cdot)\,.$$

Let $T : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}^{\mathbb{N}}$ be the left shift map defined by $T(x_0, x_1, \dots) = (x_1, x_2, \dots)$. Obviously, T preserves the measure μ. From Harris and Robbins (1953), it is known that the map T is conservative and ergodic. We consider the set $A = \{x \in \mathbb{Z}^{\mathbb{N}} : x_0 = 0\}$. As we have seen in Example 3.3.5, the set A turns out to be a Darling-Kac set with normalizing sequence $a_n = \sum_{k=1}^{n} P_0(x_k = 0)$ and hence T is a pointwise dual ergodic map.

One of the possible ways for ensuring regular variation of (a_n) is to assume

$$\sum_{k=1}^{n} P_0(\varphi \geq k) \in RV_{1-\beta} \quad \text{for some } 0 \leq \beta < 1 \,,$$

where $\varphi(x) = \min\{n \geq 1 : x_n = 0\}$, $x \in \mathbb{Z}^{\mathbb{N}}$, is the first entrance time to the set A. From Lemma 3.3 in Resnick et al. (2000), we see that $\mu(\varphi = n) = P_0(\varphi \geq n)$ and by (2.8),

$$a_n \sim \frac{1}{\Gamma(2-\beta)\Gamma(1+\beta)} \frac{n}{\mu(\varphi \leq n)} \in RV_\beta \,.$$

We will proceed to check condition (4.4). The formula on p. 156 of Aaronson (1997)) gives

$$\widehat{T}^k \mathbf{1}_{A^c \cap \{\varphi = k\}}\big(x_0, x_1, \dots\big) = \mathbf{1}_{\{x_0 = 0\}} \sum_{i_0 \neq 0} \pi_{i_0} \sum_{i_1 \neq 0} p_{i_0 i_1} \dots \sum_{i_{k-1} \neq 0} p_{i_{k-2} i_{k-1}} p_{i_{k-1} 0} \,,$$

which immediately implies (4.4).

We take a measurable function $f : \mathbb{Z}^{\mathbb{N}} \to \mathbb{R}$ that is supported by the set A and satisfies (4.5). Now, Theorem 4.2.1 applies if the parameters lie in the range $1 < \alpha < 2$, $0 \leq \beta < 1$, or $0 < \alpha \leq 1$, $0 \leq \beta < 1/(2-\alpha)$.

On the other hand, if $0 < \alpha \leq 1$ and $1/(2-\alpha) \leq \beta < 1$, we need to check the conditions given in (i) of Theorem 4.2.1. For this, we consider a two-dimensional Markov chain $((x_k, y_k),\ k \geq 0)$ with (y_k) an independent copy of (x_k). Let $P_{(i,j)}(\cdot)$ be a probability law of (x_k, y_k) starting from $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. It is now easy to check that (x_k, y_k) is also irreducible and null recurrent, and a probability measure $\mu \times \mu$ can be written as

$$(\mu \times \mu)(\cdot) = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \pi_i \pi_j P_{(i,j)}(\cdot) \,.$$

Because of Harris and Robbins (1953) again, we can say that $T \times T$ is conservative ergodic and measure preserving map on $(\mathbb{Z}^{\mathbb{N}} \times \mathbb{Z}^{\mathbb{N}}, \mathcal{B}(\mathbb{Z}^{\mathbb{N}}) \times \mathcal{B}(\mathbb{Z}^{\mathbb{N}}))$.

Evidently, the product set $A \times A$ is a Darling-Kac set. Indeed,

$$\sum_{k=1}^{n} (\widehat{T \times T})^k \mathbf{1}_{A \times A}(x,y) = \sum_{k=1}^{n} \widehat{T}^k \mathbf{1}_A(x) \widehat{T}^k \mathbf{1}_A(y) = \sum_{k=1}^{n} P_0(x_k = 0)^2 \quad \text{for } (x,y) \in A \times A\,.$$

Therefore, by the normalizing sequence $a_n' = \sum_{k=1}^n P_0(x_k = 0)^2$, the product set $A \times A$ turns out to be a Darling-Kac set, and $T \times T$ is of course pointwise dual ergodic.

Once again, by appealing to Lemma 3.3 in Resnick et al. (2000), we get

$$(\mu \times \mu)(\varphi(x,y) \le n) = \sum_{k=1}^{n} P_{(0,0)}(\varphi(x,y) \ge k) \in RV_{2(1-\beta)}.$$

Thus,

$$a_n' \sim \frac{1}{\Gamma(3-2\beta)\Gamma(2\beta)} \frac{n}{(\mu \times \mu)(\varphi(x,y) \le n)} \in RV_{2\beta - 1}.$$

To check (4.13), one more application of the formula on p. 156 of Aaronson (1997) yields

$$(\widehat{T \times T})^k \mathbf{1}_{(A \times A)^c \cap \{\varphi(x,y)=k\}} \big((x_0,y_0),(x_1,y_1)\dots\big)$$

$$= \mathbf{1}_{\{(x_0,y_0)=(0,0)\}} \sum_{(i_0,j_0) \ne (0,0)} \pi_{i_0} \pi_{j_0} \sum_{(i_1,j_1) \ne (0,0)} p_{i_0 i_1} p_{j_0 j_1} \cdots \sum_{(i_{k-1},j_{k-1}) \ne (0,0)} p_{i_{k-2} i_{k-1}} p_{i_{k-1} 0} p_{j_{k-2} j_{k-1}} p_{j_{k-1} 0}\,.$$

Therefore (4.13) holds, and in this case, Theorem 4.2.1 applies as well.

Example 4.3.2. In this example, we will, once again, consider the basic AFN-system introduced in Example 3.3.6. The setup of the system here is totally the same as that in Example 3.3.6 (one minor difference is that we will allow the limiting case $\beta = 0$ as well). Let T be a basic AFN map, and we will take the same Darling-Kac set A, which is bounded away from a family of indifferent fixed points.

Suppose that the parameters α and β lie in the range of either $1 < \alpha < 2$, $0 \le \beta < 1$, or $0 < \alpha \le 1$, $0 \le \beta < 1/(2-\alpha)$. If a measurable function $f : E \to \mathbb{R}$ is supported by the set A together with a proper integrability assumption, then Theorem 4.2.1 applies.

Suppose that $0 < \alpha \leq 1$ and $1/(2-\alpha) \leq \beta < 1$. In this case, we will check (ii) in Theorem 4.2.1, because unlike Example 4.3.1, the product map $T \times T$ is not generally conservative and ergodic. According to condition (ii), however, the Darling-Kac set A must be a uniformly returning set. Unfortunately, this is not always the case for a general basic AFN-system. To overcome this difficulty, we have to impose certain additional assumptions; see for example, Thaler (2000). If we restrict ourselves to such a type of a basic AFN-system, then (ii) is satisfied and consequently Theorem 4.2.1 follows.

Example 4.3.3. We will construct the dynamical system by a S-unimodal map with flat critical point. The main reference here is Zweimüller (2004). Let $T : [a,b] \to [a,b]$ be a S-unimodal map with flat critical point $c \in (a,b)$. That is, the Schwarzian derivative of T is nonpositive: $\mathcal{S}T = T'''/T' - \frac{3}{2}(T''/T')^2 \leq 0$, and all derivatives at the critical point c vanish: $T^{(n)}c = 0$ for all $n \geq 1$. Further, assume that $Ta = Tb = a$ and that $\int_{[a,b]} \ln |T'| d\lambda = -\infty$ (λ is the one-dimensional Lebesgue measure). In addition, we suppose that T satisfies Misiurewicz condition, i.e., there is an open interval I containing c such that $T^n c \notin I$ for all $n \geq 1$. Also, assume that there exists a positive and finite Lyapunov exponent $\lambda_c = \lim_{n\to\infty} n^{-1} \ln |(T^n)'(Tc)|$.

The dynamical effect of a flat critical point is that the closer the orbit gets to c, the slower it moves away from the critical orbit $(T^n c, n \geq 1)$. Consequently, the orbit stays in neighborhood of $(T^n c, n \geq 1)$ for a nonnegligible amount of time.

It is shown in Zweimüller (2004) that there exists an infinite measure $\mu \ll \lambda$ such that T is a conservative ergodic and measure preserving map on $([a,b], \mathcal{B}([a,b]), \mu)$.

From Zweimüller (2004) and Zweimüller (2007a), one can find a Darling-Kac set A, which is bounded away from the critical orbit $(T^n c, n \geq 1)$ such that

$$\left(\frac{\widehat{T}^n \mathbf{1}_{A\cap\{\varphi=n\}}}{\mu(A \cap \{\varphi = n\})}, n \geq 1 \right) \quad \text{is bounded on } A.$$

This property in fact proves condition (4.4). The existence of a positive and finite Lyapunov exponent guarantees that the normalizing sequence (a_n) for the Darling-Kac set is a regularly varying function of the order $0 < \beta < 1$ (see Theorem 7 in Zweimüller (2004)).

Suppose that the range of the parameters α and β is either $1 < \alpha < 2$, $0 < \beta < 1$ or $0 < \alpha \leq 1$, $0 < \beta < 1/(2-\alpha)$. If a measurable function f satisfies a proper integrability condition and is supported by the set A, Theorem 4.2.1 applies.

4.4 Appendix

Lemmas 4.4.1, 4.4.2, and 4.4.3 below are necessary components for the proof of Propositions 4.2.5 and 4.2.7. Throughout the Appendix, we suppose the conditions described in Theorem 4.2.1. The most important result established in these lemmas is

$$E\left|\frac{1}{c_n}\sum_{k=1}^{n} f_k(V_i)f_{k+h}(V_j)q(V_i)^{-1/\alpha'}q(V_j)^{-1/\alpha'}\right|^{\alpha'} \to 0, \quad \text{for } i \neq j\,. \tag{4.28}$$

The first lemma treats the case when α and β lie in the range of (4.9).

Lemma 4.4.1. *Suppose the conditions of Theorem 4.2.1, where α and β lie in the range of* (4.9). *We fix a constant $\xi > 0$ such that*

$$\xi < \alpha - 1 \quad \textit{if}\; 1 < \alpha < 2\,,$$

$$\xi < \alpha\left(1 - \frac{1}{2-\beta(2-\alpha)}\right) \quad \textit{if}\; 0 < \alpha \leq 1,\; 0 \leq \beta < \frac{1}{2-\alpha}\,.$$

Let $\alpha' = \alpha - \xi$. Then, (4.28) holds.

Proof. First, suppose that $1 < \alpha < 2$. Since $\alpha' > 1$, Minkowski's inequality applies to obtain

$$E\left|\frac{1}{c_n}\sum_{k=1}^{n} f_k(V_i)f_{k+h}(V_j)q(V_i)^{-1/\alpha'}q(V_j)^{-1/\alpha'}\right|^{\alpha'} = \frac{1}{c_n^{\alpha'}}\int_{E\times E}\Big|\sum_{k=1}^{n} f_k(x)f_k(y)\Big|^{\alpha'}(\mu\times\mu)(dx\;dy)$$

$$\leq \left(\frac{n}{c_n}\right)^{\alpha'} \left(\int_E |f|^{\alpha'} d\mu\right)^2 .$$

Since $n/c_n \in RV_{(1-\beta)(1-2/\alpha)}$ with $(1-\beta)(1-2/\alpha) < 0$, we have $n/c_n \to 0$.

Next, suppose that $0 < \alpha \leq 1$ and $0 \leq \beta < 1/(2-\alpha)$. In this case, a simple application of the triangle inequality gives

$$E\left|\frac{1}{c_n}\sum_{k=1}^{n} f_k(V_i)f_{k+h}(V_j)q(V_i)^{-1/\alpha'}q(V_j)^{-1/\alpha'}\right|^{\alpha'} = \frac{1}{c_n^{\alpha'}}\int_{E\times E}\Big|\sum_{k=1}^{n} f_k(x)f_k(y)\Big|^{\alpha'}(\mu\times\mu)(dx\,dy)$$

$$\leq \frac{n}{c_n^{\alpha'}}\left(\int_E |f|^{\alpha'} d\mu\right)^2 .$$

However, we see that $n/c_n^{\alpha'} \in RV_{1-\alpha'(\beta+2(1-\beta)/\alpha)}$ with $1-\alpha'(\beta+2(1-\beta)/\alpha) < 0$. $\square$

As compared with the case where α and β lie in the range (4.9), establishing (4.28) in the case $0 < \alpha \leq 1$ and $1/(2-\alpha) \leq \beta < 1$ is more difficult. Indeed, if $0 < \alpha \leq 1$ and $1/(2-\alpha) \leq \beta < 1$, a simple manipulation of the basic inequalities as we have seen in Lemma 4.4.1 cannot lead us to (4.28). Thus, we need some alternative approaches.

A possible alternative approach is to assume that the product map $T \times T$ still has *nice properties* as given in (i) of Theorem 4.2.1. Specifically, in the next lemma, we will assume that $T \times T$ is still conservative and ergodic on a measure space $(E \times E, \mathcal{E} \times \mathcal{E}, \mu \times \mu)$, and further, it is also pointwise dual ergodic. The benefit of this assumption is that we can explicitly calculate the exact growth rate of the quantity $\int_{E\times E} |\sum_{k=1}^{n} f_k(x)f_k(y)|^{\alpha'}(\mu\times\mu)(dx\,dy)$.

Lemma 4.4.2. *Suppose the conditions of Theorem 4.2.1, where* $0 < \alpha \leq 1$ *and* $1/(2-\alpha) \leq \beta < 1$, *and particularly assume condition* (i). *We fix* $0 < \xi < \alpha^2/(\alpha+2)$ *and let* $\alpha' = \alpha - \xi$. *Then, (4.28) follows.*

Proof. Denote by $S_n(f\times f)(x,y) = \sum_{k=1}^{n} f_k(x)f_k(y)$ a partial sum defined on a product space $E \times E$. By virtue of (4.13), proceeding as in the proof of Proposition 2.2.5, we can get

$$\frac{S_n(f\times f)(x,y)}{a_n'} \Rightarrow \mu(f)^2\Gamma(2\beta)M_{2\beta-1}(1-V_{2\beta-1}) \quad \text{in } \mathbb{R}\,, \tag{4.29}$$

where the weak convergence takes place under a probability measure

$$(\mu \times \mu)_n(\cdot) = (\mu \times \mu)(\cdot \cap \{\varphi(x,y) \le n\})/(\mu \times \mu)(\varphi(x,y) \le n)\,. \tag{4.30}$$

Here, $M_{2\beta-1}(t)$ is the Mittag-Leffler process with exponent $2\beta - 1$, and $V_{2\beta-1}$ is defined by (2.34).

From (2.6), (2.8), and the assumption that $T \times T$ is a conservative and ergodic map, we can obtain

$$(\mu \times \mu)(\varphi(x,y) \le n) \sim \frac{1}{\Gamma(3-2\beta)\Gamma(2\beta)} \frac{n}{a_n'}\,,$$

from which $(\mu \times \mu)(\varphi(x,y) \le n) \in RV_{2(1-\beta)}$ follows.

Now, we have

$$E\Big|\sum_{k=1}^{n} f_k(V_i) f_{k+h}(V_j) q(V_i)^{-1/\alpha'} q(V_j)^{-1/\alpha'}\Big|^{\alpha'} = \int_{E\times E} |S_n(f \times f)(x,y)|^{\alpha'} (\mu \times \mu)(dx\, dy)$$

$$= (a_n')^{\alpha'} (\mu \times \mu)(\varphi(x,y) \le n) \int_{E\times E} \left|\frac{S_n(f \times f)(x,y)}{a_n'}\right|^{\alpha'} (\mu \times \mu)_n(dx\, dy)\,.$$

Uniform integrability of $(|S_n(f \times f)/a_n'|^{\alpha'}, n \ge 1)$ and (4.29) guarantee

$$\int_{E\times E} \left|\frac{S_n(f \times f)(x,y)}{a_n'}\right|^{\alpha'} (\mu \times \mu)_n(dx\, dy) \to \mu(f)^{2\alpha'} \Gamma(2\beta)^{\alpha'} E M_{2\beta-1}(1 - V_{2\beta-1})^{\alpha'} < \infty\,.$$

On the other hand, (4.8) implies $c_n^{\alpha'} \in RV_{\alpha'(\beta+2(1-\beta)/\alpha)}$. Thus,

$$E\left|\frac{1}{c_n}\sum_{k=1}^{n} f_k(V_i) f_{k+h}(V_j) q(V_i)^{-1/\alpha'} q(V_j)^{-1/\alpha'}\right|^{\alpha'} \in RV_{(2\beta-1)\alpha' + 2(1-\beta) - \alpha'(\beta+2(1-\beta)/\alpha))}.$$

Owing to the constraint on ξ, we have $(2\beta-1)\alpha' + 2(1-\beta) - \alpha'(\beta + 2(1-\beta)/\alpha) < 0$ and hence (4.28) is complete. □

The argument in Lemma 4.4.2 requires that the product map $T \times T$ to be conservative and ergodic. However, this is not necessarily true in general; see Example 4.3.2. It is, therefore, beneficial to get (4.28) without conservativity and ergodicity of the product map $T \times T$.

Lemma 4.4.3. *Suppose the conditions of Theorem 4.2.1, where $0 < \alpha \le 1$ and $1/(2-\alpha) \le \beta < 1$, and particularly assume condition (ii). We fix $0 < \xi < \alpha^2/(\alpha+2)$ and let $\alpha' = \alpha - \xi$. Then, (4.28) follows.*

Proof. We start by claiming

$$\frac{1}{a'_n}\sum_{k=1}^{n}(\widehat{T\times T})^k \mathbf{1}_{A\times A}(x,y) \to \mu(A)^2 \quad \text{uniformly, a.e. on } A\times A\,, \tag{4.31}$$

where

$$a'_n = \left(\frac{\Gamma(1+\beta)}{\Gamma(\beta)}\right)^2 \frac{\Gamma(2\beta-1)}{\Gamma(2\beta)}\frac{a_n^2}{n}\,.$$

Indeed, from (2.9) and (4.14), we see that

$$\sum_{k=1}^{n}(\widehat{T\times T})^k \mathbf{1}_{A\times A}(x,y) = \sum_{k=1}^{n}\widehat{T}^k\mathbf{1}_A(x)\widehat{T}^k\mathbf{1}_A(y)$$

$$\sim \frac{\mu(A)^2}{\Gamma(\beta)^2\Gamma(2-\beta)^2}\sum_{k=1}^{n}\frac{1}{w_k^2} \quad \text{uniformly, a.e. on } A\times A\,.$$

Applying the Karamata's Tauberian theorem for power series (see e.g., Corollary 1.7.3 in Bingham et al. (1987)) to relation (2.8),

$$\sum_{k=1}^{n}\frac{1}{w_k^2} \sim \Gamma(2-\beta)^2\Gamma(1+\beta)^2\frac{\Gamma(2\beta-1)}{\Gamma(2\beta)}\frac{a_n^2}{n} \quad \text{as } n\to\infty\,.$$

Thus, (4.31) is complete.

Now, (4.31) ensures that $A\times A$ can be viewed as a Darling-Kac set for the product map $T\times T$. As argued in Remark 2.2.6, even if $T\times T$ is neither conservative nor ergodic,

$$\int_{E\times E}\left(\frac{S_n(\mathbf{1}_{A\times A})(x,y)}{a'_n}\right)^r (\mu\times\mu)(dx\,dy) \sim \mu(A)^{2r}r!\frac{\Gamma(2\beta)^{r-1}}{\Gamma(r(2\beta-1)+3-2\beta)}\frac{n}{a'_n}$$

holds.

Next, we define a probability measure $(\mu\times\mu)_n(\cdot)$ by

$$(\mu\times\mu)_n(\cdot) = (\mu\times\mu)(\{\varphi(x)\le n, \varphi(y)\le n\}\cap\cdot)/\mu(\varphi\le n)^2.$$

Notice that the above definition of $(\mu\times\mu)_n$ differs from (4.30) given in Lemma 4.4.2. Then, we have

$$\int_{E\times E}\left(\frac{S_n(\mathbf{1}_{A\times A})(x,y)}{a'_n}\right)^r (\mu\times\mu)_n(dx\,dy)$$

$$= \frac{1}{\mu(\varphi\le n)^2}\int_{E\times E}\left(\frac{S_n(\mathbf{1}_{A\times A})(x,y)}{a'_n}\right)^r (\mu\times\mu)(dx\,dy)$$

$$\to \mu(A)^{2r} r! \frac{\Gamma(\beta)^2 \Gamma(2-\beta)^2 \Gamma(2\beta)^r}{\Gamma(2\beta-1)\Gamma(r(2\beta-1)+3-2\beta)} \equiv \mu(A)^{2r} \eta_r \,.$$

The sequence (η_r) determines, uniquely in law, a random variable Z_β, whose rth moment coincides with η_r itself. To see this, it is enough to check the Carleman sufficient condition $\sum_{k=1}^\infty \eta_{2k}^{-1/2k} = \infty$, which can be easily checked by Stirling's formula together with elementary algebra. It is thus concluded that with respect to $(\mu \times \mu)_n$,

$$\frac{S_n(\mathbf{1}_{A\times A})(x,y)}{a_n'} \Rightarrow \mu(A)^2 Z_\beta \quad \text{in } \mathbb{R} \,.$$

Since f is a bounded and is supported by A, there is a constant $C_1 > 0$ such that

$$E\Big|\sum_{k=1}^n f_k(V_i) f_{k+h}(V_j) q(V_i)^{-1/\alpha'} q(V_j)^{-1/\alpha'}\Big|^{\alpha'} = \int_{E\times E} |S_n(f \times f)(x,y)|^{\alpha'} (\mu\times\mu)(dx\, dy)$$

$$\le C_1 \int_{E\times E} |S_n(\mathbf{1}_{A\times A})(x,y)|^{\alpha'} (\mu\times\mu)(dx\, dy)$$

$$= C_1 (a_n')^{\alpha'} \mu(\varphi \le n)^2 \int_{E\times E} \left| \frac{S_n(\mathbf{1}_{A\times A})(x,y)}{a_n'} \right|^{\alpha'} (\mu\times\mu)_n(dx\, dy) \,.$$

Because of the uniform integrability of $(|S_n(\mathbf{1}_{A\times A})/a_n'|^{\alpha'}, n \ge 1)$, we see that $\int_{E\times E} |S_n(\mathbf{1}_{A\times A})/a_n'|^{\alpha'} \, d(\mu\times\mu)_n$ converges to some positive finite constant. The rest of the discussion is the same as Lemma 4.4.2. □

Finally, we provide useful inequalities, which will supplementarily be used in the proof of Propositions 4.2.5 and 4.2.7.

Lemma 4.4.4. *Fix $\xi > 0$ as specified in Lemma 4.4.1, 4.4.2, or 4.4.3. Let $\alpha' = \alpha - \xi$ and define*

$$W_{ij}^{(n,\alpha')} = \frac{1}{c_n} \sum_{k=1}^n f_k(V_i) f_{k+h}(V_j) q(V_i)^{-1/\alpha'} q(V_j)^{-1/\alpha'} \,.$$

Let

$$\ln_+ x = \begin{cases} \ln x & \text{if } x > 1 \,, \\ 0 & \text{otherwise.} \end{cases}$$

(a) There exist an integer $m_0 > 0$ and constants $C > 0$, $\gamma < \alpha'$, such that for any $m \ge m_0$,

$$E\left| \sum_{m<i<j<\infty} \epsilon_i \epsilon_j U_{\alpha,1}^{\leftarrow}\left(\frac{\Gamma_i q(V_i)}{2}\right) U_{\alpha,1}^{\leftarrow}\left(\frac{\Gamma_j q(V_j)}{2}\right) \frac{1}{c_n} \sum_{k=1}^n f_k(V_i) f_{k+h}(V_j) \mathbf{1}_{\{|W_{ij}^{(n,\alpha')}|^{\alpha'} \le ij\}} \right|^{\alpha'}$$

$$\leq C\Big(E\big(|W_{ij}^{(n,\alpha')}|^{\alpha'}(1+\ln_+|W_{ij}^{(n,\alpha')}|)\big)\Big)^{\gamma},$$

$$E\left|\sum_{m<i<j<\infty}\epsilon_i\epsilon_j U_{\alpha,1}^{\leftarrow}\left(\frac{\Gamma_i q(V_i)}{2}\right)U_{\alpha,1}^{\leftarrow}\left(\frac{\Gamma_j q(V_j)}{2}\right)\frac{1}{c_n}\sum_{k=1}^{n} f_k(V_i)f_{k+h}(V_j)\mathbf{1}_{\{|W_{ij}^{(n,\alpha')}|^{\alpha'}>ij\}}\right|^{\alpha'}$$
$$\leq CE\big(|W_{ij}^{(n,\alpha')}|^{\alpha'}(1+\ln_+^2|W_{ij}^{(n,\alpha')}|)\big).$$

(b) There exist an integer $m_0>0$ and constants $C>0$, $\gamma<\alpha'$, such that for any $m\geq m_0$ and $i\geq 1$,

$$E\left|\sum_{j=m+1}^{\infty}\epsilon_j U_{\alpha,1}^{\leftarrow}\left(\frac{\Gamma_j q(V_j)}{2}\right)\frac{1}{c_n}\sum_{k=1}^{n} f_k(V_i)f_{k+h}(V_j)q(V_i)^{-1/\alpha'}\mathbf{1}_{\{|W_{ij}^{(n,\alpha')}|^{\alpha'}\leq j\}}\right|^{\alpha'}$$
$$\leq C\big(E|W_{ij}^{(n,\alpha')}|^{\alpha'}\big)^{\gamma},$$

$$E\left|\sum_{j=m+1}^{\infty}\epsilon_j U_{\alpha,1}^{\leftarrow}\left(\frac{\Gamma_j q(V_j)}{2}\right)\frac{1}{c_n}\sum_{k=1}^{n} f_k(V_i)f_{k+h}(V_j)q(V_i)^{-1/\alpha'}\mathbf{1}_{\{|W_{ij}^{(n,\alpha')}|^{\alpha'}>j\}}\right|^{\alpha'}$$
$$\leq CE\big(|W_{ij}^{(n,\alpha')}|^{\alpha'}(1+\ln_+|W_{ij}^{(n,\alpha')}|)\big).$$

Proof. The proof is analogous to that of Proposition 5.1 in Samorodnitsky and Szulga (1989), but an obvious upper bound $U_{\alpha,1}^{\leftarrow}(x)\leq Cx^{-1/\alpha'}$, $x>0$, has to be suitably applied. □

Remark 4.4.5. The inequalities in Lemma 4.4.4 will still hold, even if the parameter α' and the inverse function $U_{\alpha,1}^{\leftarrow}(\cdot)$ are replaced by the constant p_0 given in (4.3) and $U_{\alpha,2}^{\leftarrow}(\cdot)$, respectively.

CHAPTER 5

LIMIT THEORY FOR PARTIAL MAXIMA FOR SYMMETRIC α-STABLE PROCESSES GENERATED BY CONSERVATIVE FLOWS

5.1 The setup

We will consider a stationary symmetric α-stable process $\mathbf{X} = (X_1, X_2, \dots)$ with $0 < \alpha < 2$ (we call it a SαS process for short). Every (not necessarily stationary) SαS process has an integral representation

$$X_n = \int_E f_n(x) dM(x), \quad n = 1, 2, \dots \tag{5.1}$$

for an SαS random measure M on a measurable space $(E, \mathcal{E})$ with σ-finite control measure μ and a family $(f_n) \subseteq L^\alpha(\mu)$. Bretagnolle et al. (1966) and Schreiber (1972) originally studied a general integral representation of such types and even treated SαS processes $(X(t),\, t \in T)$ with an arbitrary index set T. If, additionally, the process $\mathbf{X} = (X_1, X_2, \dots)$ is stationary, one can choose the kernel (f_n) in the canonical form

$$f_n(x) = a_n(x) \left(\frac{d\mu \circ \phi^n}{d\mu}(x) \right)^{1/\alpha} f \circ \phi^n(x), \quad x \in E \tag{5.2}$$

for $n = 1, 2, \dots$, where $\phi : E \to E$ is a one-to-one map with both ϕ and ϕ^{-1} measurable, mapping the control measure μ into an equivalent measure, and

$$a_n(x) = \prod_{j=0}^{n-1} u \circ \phi^j(x), \quad x \in E$$

for $n = 1, 2, \dots$, with $u : E \to \{-1, 1\}$ a measurable function and $f \in L^\alpha(\mu)$. We refer to Rosiński (1995) for more details, and to Samorodnitsky and Taqqu (1994) for general information on the integrals by SαS random measures.

In this chapter, we will focus on a little more restrictive class of stationary SαS processes

$$X_n = \int_E f \circ T^n(x) dM(x), \quad n = 1, 2, \dots. \tag{5.3}$$

Here M is an SαS random measure on a measurable space $(E, \mathcal{E})$ with $0 < \alpha < 2$ and σ-finite infinite control measure μ. $T : E \to E$ is a measurable map that preserves μ, and $f \in L^\alpha(\mu)$. We denote $f_n(x) = f \circ T^n(x)$, $x \in E$, $n = 1, 2, \dots$.

Adopting the single Poisson jump assumption as in Samorodnitsky (2004), we will investigate the limit behavior of the partial maxima $\max_{1 \le k \le \lfloor nt \rfloor} |X_k|$, $n = 1, 2, \dots, t \ge 0$. A central assumption of this chapter is, once again, that the flow (T^n) is a conservative flow. Namely, $T : E \to E$ is a conservative ergodic and measure preserving map on a σ-finite infinite measure space $(E, \mathcal{E}, \mu)$. We will assume that T is a pointwise dual ergodic map with regularly varying return sequence $(a_n) \in RV_{1-\beta}$, $1/2 < \beta < 1$. In the previous chapters, we have set $(a_n) \in RV_\beta$, but for notational convenience, we will switch the exponent from β to $1 - \beta$. Due to pointwise dual ergodicity for the operator T, one can always find a uniform set $A \in \mathcal{E}$, $0 < \mu(A) < \infty$, under which (2.3) holds. Moreover, as in Chapter 3, we will add an extra assumption on the set A; let $A_0 = A$, $A_n = A^c \cap \{\varphi = n\}$, $n \ge 1$, with φ being the first entrance time of A. There exists a measurable function $K : E \to \mathbb{R}_+$ such that

$$\frac{\widehat{T}^n \mathbf{1}_{A_n}}{\mu(A_n)} \to K \quad \text{uniformly, a.e. on } A. \tag{5.4}$$

We will further assume that f is supported by A, and it satisfies

$$\sup_{n \ge 1} w_n^{-1} \int_E \max_{1 \le k \le n} |f_k(x)|^{\alpha + \xi} \mu(dx) < \infty \tag{5.5}$$

for some $\xi > 0$ and a wandering sequence (w_n).

As in Chapters 3 and 4, the marginal heavy tailedness and long memory in the process $\mathbf{X} = (X_1, X_2, \dots)$ will completely depict the limiting behavior of the partial maxima process. In particular, the assumption that the process $\mathbf{X}$ is generated by a conservative flow relates to the long memory in the process.

5.2 Limiting Processes

Before stating the main limit theorem, we need to identify the limiting process. We define a σ-finite and infinite measure

$$\nu_\beta(dx) = \beta x^{\beta-1}dx, \quad 0 < x < \infty \ \text{ for some } \beta \in (0,1)\,.$$

Let $0 < \alpha < 2$ and $M_{\alpha,\beta}$ be an independently scattered, α-Fréchet sup-measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ with control measure ν_β. Specifically, $M_{\alpha,\beta}$ is formulated as follows:

- For any $0 = t_0 < t_1 < \cdots < t_d < \infty$, $\big(M_{\alpha,\beta}((t_{i-1}, t_i]),\, i = 1, \ldots, d\big)$ are independent random variables.
- For any $t > 0$, we have

  $$P\big(M_{\alpha,\beta}((0,t]) \le x\big) = \exp\{-\nu_\beta((0,t])x^{-\alpha}\} = \exp\{-t^\beta x^{-\alpha}\}, \quad x > 0\,,$$

  that is, $M_{\alpha,\beta}((0,t])$ is an α-Fréchet random variable with scale coefficient $t^{\beta/\alpha}$.
- For any $0 = t_0 < t_1 < \cdots < t_d < \infty$,

  $$M_{\alpha,\beta}((0,t_d]) = \bigvee_{j=1}^{d} M_{\alpha,\beta}((t_{j-1}, t_j]) \equiv \max_{1\le j\le d} M_{\alpha,\beta}((t_{j-1}, t_j]) \quad \text{a.s.}\,.$$

In fact, $Z_{\alpha,\beta}(t) = M_{\alpha,\beta}((0,t])$ will turn out to be a weak limit in the main limit theorem. $(Z_{\alpha,\beta}(t))$ has almost surely nondecreasing, right-continuous sample paths with left limits. $(Z_{\alpha,\beta}(t))$ is, in fact, a new class of an α-Fréchet process characterized by a parameter $\beta \in (0,1)$. A formal substitution of $\beta = 1$ leads to the so-called classical extremal process (see Chapter 4 in Resnick (1987)). It is also straightforward to check that $(Z_{\alpha,\beta}(t),\, t \ge 0)$ is self-similar with exponent $H = \beta/\alpha$; let $\bigwedge_{i=1}^d a_i = \min_{1\le i\le d} a_i$. For any $0 = t_0 < t_1 < \cdots < t_d < \infty$, $c > 0$, and $\lambda_i > 0$, $i = 1, \ldots, d$, we have

$$P\big(Z_{\alpha,\beta}(ct_i) \le \lambda_i,\ \ i = 1, \ldots, d\big) = P\left(M_{\alpha,\beta}((ct_{i-1}, ct_i]) \le \bigwedge_{j=i}^{d} \lambda_j\,,\ \ i = 1, \ldots, d\right)$$

$$= \prod_{i=1}^{d} P\left(M_{\alpha,\beta}((ct_{i-1}, ct_i]) \le \bigwedge_{j=i}^{d} \lambda_j \right) = \exp\left\{ -\sum_{i=1}^{d} \big((ct_i)^\beta - (ct_{i-1})^\beta\big) \left(\bigwedge_{j=i}^{d} \lambda_j \right)^{-\alpha} \right\}$$
$$= P\big(c^{\beta/\alpha} Z_{\alpha,\beta}(t_i) \le \lambda_i,\ \ i = 1, \ldots, d\big)\,.$$

Moreover, we can prove that $\big(Z_{\alpha,\beta}(t)\big)$ possesses the following property.

Definition 5.2.1. *A stochastic process $(Y(t),\, t \ge 0)$ has max-stationary increments if for every $r > 0$, there exists a stochastic process $\big(Y^{(r)}(t),\, t \ge 0\big)$ such that*

$$\big(Y^{(r)}(t),\, t \ge 0\big) \stackrel{d}{=} (Y(t),\, t \ge 0)\,,$$
$$(Y(t+r),\, t \ge 0) \stackrel{d}{=} \big(Y(r) \vee Y^{(r)}(t),\, t \ge 0\big)\,.$$

In the case of $(Z_{\alpha,\beta}(t))$, observe first that $(Z_{\alpha,\beta}(t))$ is distributionally equal to

$$U_{\alpha,\beta}(t) = \sup\{\, j_k : t_k \le t^\beta \}\,,\ \ t \ge 0\,,$$

where (j_k, t_k) denotes the points of a Poisson random measure on $\mathbb{R}_+^2$ with mean measure $\rho_* \times \lambda$, $\rho_*(x, \infty) = x^{-\alpha}$, $x > 0$ and λ is the one-dimensional Lebesgue measure on $\mathbb{R}_+$.

Given $r > 0$, we define

$$U^{(r)}_{\alpha,\beta}(t) = \sup\{\, j_k : (t+r)^\beta - t^\beta \le t_k \le (t+r)^\beta \}\,.$$

Then, it is not difficult to show that

$$\big(U^{(r)}_{\alpha,\beta}(t),\, t \ge 0\big) \stackrel{d}{=} \big(U_{\alpha,\beta}(t),\, t \ge 0\big)\,,$$
$$U_{\alpha,\beta}(t+r) = U_{\alpha,\beta}(r) \vee U^{(r)}_{\alpha,\beta}(t) \ \text{ for all } t \ge 0;$$

thus, $(Z_{\alpha,\beta}(t))$ possesses max-stationary increments.

Under a mild condition, whenever the normalized partial maxima process weakly converges to some nondegenerate weak limit, the limit must be a self-similar process (Lamperti (1962)). Hence, a study of self-similar α-Fréchet processes with max-stationary increments may provide deeper understanding of general weak limits of the normalized partial maxima. Unfortunately, for lack of space, we leave the details to future work.

Finally, we recall a certain useful spectral representation for $(Z_{\alpha,\beta}(t))$ when t is restricted in $[0,1]$:

$$(Z_{\alpha,\beta}(t),\, 0 \le t \le 1) \overset{d}{=} \left(\bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \mathbf{1}_{\{V_j \le t\}},\, 0 \le t \le 1 \right),$$

where Γ_j, $j = 1, 2, \ldots$, are arrival times of a unit rate Poisson process, and (V_j) are i.i.d. random variables with $P(V_1 \le x) = x^\beta$, $0 < x \le 1$, independent of (Γ_j). We refer to Stoev and Taqqu (2005) for more information about α-Fréchet sup-measures and their spectral representation.

5.3 Limit Theorem on the Partial Maxima

This section describes the limit theorems on the partial maxima $\max_{1 \le k \le \lceil nt \rceil} |X_k|$, $t \ge 0$, for the process $\mathbf{X} = (X_1, X_2, \ldots)$ given in (5.3). As we did in Chapters 3 and 4, we will first find how rapidly the partial maxima grow. Indeed, we will later justify that

$$b_n = \left(\int_E \max_{1 \le k \le n} |f_k(x)|^\alpha \mu(dx) \right)^{1/\alpha}, \quad n = 1, 2, \ldots$$

works as a proper normalizing sequence.

The quantity b_n itself has been known to capture, to some extent, the growing rate of $\max_{1 \le k \le n} |X_k|$, even without the stationarity of the process $\mathbf{X}$; see Marcus (1984). According to Samorodnitsky (2004), if a stationary SαS process is given in a general form (5.1) together with a canonical kernel (5.2), and if the process is generated by a conservative flow, then the sequence (b_n) grows strictly slower than $n^{1/\alpha}$. Given the more restrictive form (5.3), we will see that under the conditions of the main limit theorem,

$$(b_n) \in RV_{\beta/\alpha}\,. \tag{5.6}$$

This implies that the growing rate of the partial maxima is determined by not only heavy tailedness of the marginals but also the memory length. This is in contrast to processes generated by dissipative flows. For example, the partial maxima for moving averages with regularly varying innovations of index $-\alpha$ grow at a regularly varying rate of exponent $1/\alpha$. Substituting $\beta = 1$ into (5.6) yields $(b_n) \in RV_{1/\alpha}$, which means that as β gets closer to 1, the process exhibits shorter memory. Equivalently, smaller β corresponds to longer memory in the process.

Clearly, the weak convergence of $b_n^{-1} \max_{1 \le k \le \lfloor nt \rfloor} |X_k|$ should be explored in $D[0,\infty)$ (the space of right-continuous functions with left limits). In this chapter, we shall endow the space $D[0,\infty)$ with two different topologies that are called the Skorohod J_1-topology and the Skorohod M_1-topology. As their names show, the study of these topologies was originally started by Skorohod (1956). The J_1-topology is more famous and is rigorously reviewed by, for example, Billingsley (1999). To understand the difference between these two topologies, we consider a sequence (x_n) in $D[0,1]$ whose limit is x. To obtain $x_n \to x$ in the J_1-topology, x_n must have a single jump around the jump of x and, further, both the magnitude and location of the jump in x_n must converge to those of the limit x. For this reason, when we equip $D[0,1]$ with the J_1-topology, neither

$$x_n = n(t - 2^{-1} + n^{-1})\mathbf{1}_{[2^{-1}-n^{-1},2^{-1})} + \mathbf{1}_{[2^{-1},1]} \tag{5.7}$$

nor

$$x_n = 2^{-1}\mathbf{1}_{[2^{-1}-n^{-1},2^{-1})} + \mathbf{1}_{[2^{-1},1]} \tag{5.8}$$

converges to $x = \mathbf{1}_{[2^{-1},1]}$ (these examples are provided in Chapter 3 of Whitt (2002)). In the case of the M_1-topology, (x_n) is allowed to have multiple jumps around a single jump of x. However, the completed graph (graph + vertical segments) of (x_n) must approach that of x in the sense of the Skorohod M_1-metric. In fact, the sequences (5.7) and (5.8) converge to $x = \mathbf{1}_{[2^{-1},1]}$ under the Skorohod M_1-metric. Whitt (2002) gives an elegant review of these Skorohod topologies.

In Theorem 5.3.1 below, if a marginal of the process $\mathbf{X}$ has the simplest integrand $f = \mathbf{1}_A$, then $b_n^{-1} \max_{1\le k\le \lfloor nt \rfloor} |X_k|$ possesses "single jump structure"; therefore, weak convergence occurs in the J_1-topology. In general, however, $b_n^{-1} \max_{1\le k\le \lfloor nt \rfloor} |X_k|$ will have "multiple jump structure," and hence the space $D[0,\infty)$ must be endowed with the M_1-topology.

Finally, we recall that C_α denotes an α-stable tail constant; see (3.18), and now, we are ready to state the main limit theorem.

Theorem 5.3.1. *Let T be a conservative ergodic and measure preserving map on a σ-finite infinite measure space $(E, \mathcal{E}, \mu)$. We assume that T is a pointwise dual ergodic map with normalizing sequence $(a_n) \in RV_{1-\beta}$, $1/2 < \beta < 1$. Suppose that T admits a uniform set $A \in \mathcal{E}$, $0 < \mu(A) < \infty$.*
Let M be a SαS random measure, $0 < \alpha < 2$ on $(E, \mathcal{E})$ with control measure μ.
Let $f : E \to \mathbb{R}$ be a $L^\alpha(\mu)$-integrable function that is supported by A and satisfies (5.5).
Then,

$$b_n = \left(\int_E \max_{1\le k\le n} |f_k(x)|^\alpha \mu(dx) \right)^{1/\alpha} \in RV_{\beta/\alpha}\,.$$

Furthermore, if T satisfies (5.4), the stationary SαS process $\mathbf{X}$ given in (5.3) satisfies

$$\frac{1}{b_n} \max_{1\le k\le \lfloor nt \rfloor} |X_k| \Rightarrow C_\alpha^{1/\alpha} Z_{\alpha,\beta}(t) \quad \text{in } D[0,\infty)\,,$$

where $\Rightarrow$ means weak convergence and the space $D[0,\infty)$ is equipped with the Skorohod M_1-topology.
Moreover, if $f = \mathbf{1}_A$, then the above convergence occurs in the Skorohod J_1-topology.

Proof. First, we claim that condition (5.5) is equivalent to the boundedness of f. If f is a bounded function, (5.5) is obviously true. Conversely, suppose for a contradiction that (5.5) holds and f is not bounded. Then,

$$B_h = \{x \in A : |f(x)| > h\}$$

is a set of positive μ-measure for every $h > 0$.

We see that

$$w_n^{-1} \int_E \max_{1 \le k \le n} |f_k(x)|^{\alpha+\xi} \mu(dx) \ge h^{\alpha+\xi} w_n^{-1} \mu\Big(\bigcup_{j=1}^{n} T^{-j} B_h\Big) .$$

Since B_h is contained in A and $\mu(B_h) > 0$, B_h is also a uniform set with the same normalizing sequence (a_n). By Proposition 3.8.7 in Aaronson (1997),

$$\mu\Big(\bigcup_{j=1}^{n} T^{-j} B_h\Big) \sim \frac{1}{\Gamma(2-\beta)\Gamma(1+\beta)} \frac{n}{a_n}$$

and so, we obtain $\mu\big(\bigcup_{j=1}^{n} T^{-j} B_h\big) \sim w_n$ as $n \to \infty$.

Hence, we have

$$\sup_{n \ge 1} w_n^{-1} \int_E \max_{1 \le k \le n} |f_k(x)|^{\alpha+\xi} \mu(dx) \ge h^{\alpha+\xi}$$

for every $h > 0$. Letting $h \to \infty$ on the right hand side yields a contradiction.

Now, we may assume that

$$\|f\|_\infty = \inf \big\{ M : |f(x)| \le M \text{ a.e. on } E \big\} < \infty .$$

Next, we will identify the regular variation exponent of (b_n).

On the one hand,

$$b_n^\alpha \le \|f\|_\infty \mu(\varphi \le n) \in RV_\beta ,$$

where $\varphi(x) = \min\{n \ge 1 : T^n x \in A\}$ is the first entrance time of the set A.

Let $\epsilon \in (0, \|f\|_\infty)$ be an arbitrary constant and introduce the event

$$B_\epsilon = \big\{ x \in A : |f(x)| \ge \|f\|_\infty - \epsilon \big\} .$$

Then, $\mu(B_\epsilon) > 0$ and thus B_ϵ is another uniform set whose normalizing sequence is given by (a_n). The lower bound for b_n^α is obtained by

$$b_n^\alpha \ge \big(\|f\|_\infty - \epsilon\big)\, \mu\Big(\bigcup_{j=1}^{n} T^{-j} B_\epsilon\Big) .$$

Since ϵ is arbitrary, the fact that $\mu\big(\bigcup_{j=1}^{n} T^{-j} B_\epsilon\big) \sim \mu(\varphi \le n)$ proves $(b_n) \in RV_{\beta/\alpha}$.

Here, it is noteworthy that if T satisfies an extra condition (5.4), we can capture asymptotic behavior of (b_n) more precisely. Let

$$M_n(f)(x) = \max_{1\le k\le n} |f_k(x)| \quad \text{and} \quad M_\infty(f)(x) = \sup_{k\ge 1} |f_k(x)|, \ \ x \in E\,.$$

Define a probability measure on E by

$$\mu_n(\cdot) = \mu(\cdot \cap \{\varphi \le n\})/\mu(\varphi \le n)\,.$$

Proposition 2.2.7 gives weak convergence

$$\mu_n \circ (M_n(f))^{-1} \Rightarrow \eta \circ (M_\infty(f))^{-1} \quad \text{in } \mathbb{R}_+\,, \tag{5.9}$$

where $\eta(\cdot) = \int_\cdot K(x)\mu(dx)$ is a probability measure on E.
Condition (5.5) leads to uniform integrability of $\big(M_n(f)^\alpha,\, n \ge 1\big)$ with respect to μ_n and, hence, (5.9) implies

$$\int_E M_n(f)^\alpha d\mu_n \to \int_A M_\infty(f)^\alpha d\eta\,.$$

Notice that the limit is finite because $\|f\|_\infty < \infty$. Thus,

$$b_n^\alpha = \mu(\varphi \le n)\int_E M_n(f)^\alpha d\mu_n \sim \mu(\varphi \le n)\int_A M_\infty(f)^\alpha d\eta\,. \tag{5.10}$$

For any random element in $D[0,\infty)$ with nondecreasing sample paths, weak convergence in the M_1-topology is equivalent to the corresponding finite-dimensional weak convergence. This equivalence can be directly checked by the tightness condition in Skorohod's original work (see Skorohod (1956)). Refer also to Avram and Taqqu (1992). Obviously, $b_n^{-1}\max_{1\le k\le \lfloor nt\rfloor}|X_k|$ has nondecreasing and right-continuous sample paths with left limits. Therefore, it is enough to show weak convergence of finite-dimensional distributions. Fix $0 = t_0 < t_1 < \cdots < t_d$, $d \ge 1$. By virtue of the regular variation of (b_n), we may and will assume that $t_d \le 1$. We use a series representation of the random vector $(X_1,\dots,X_n)$:

$$(X_k,\, k = 1,\dots,n) \stackrel{d}{=} \left(b_n C_\alpha^{1/\alpha} \sum_{j=1}^\infty \epsilon_j \Gamma_j^{-1/\alpha} \frac{f_k(U_j^{(n)})}{\max_{1\le i\le n}|f_i(U_j^{(n)})|},\, k = 1,\dots,n \right)\,, \tag{5.11}$$

where (ϵ_j) are i.i.d. Rademacher random variables taking ± 1 with probability $1/2$, (Γ_j) are arrival times of a Poisson process with unit parameter, and $(U_j^{(n)})$ are i.i.d. random variables with common distribution given by

$$\frac{d\eta_n}{d\mu}(x) = \frac{1}{b_n^\alpha} \max_{1 \le k \le n} |f_k(x)|^\alpha \,, \quad x \in E \,.$$

Here (ϵ_j), (Γ_j), and $(U_j^{(n)})$ are taken to be independent. We refer to Section 3.10 of Samorodnitsky and Taqqu (1994) for series representation of α-stable random vectors.

An important observation is that, in the series representation (5.11), only one largest Poisson jump will contribute to the value of the maximum. Specifically, we may say that as $n \to \infty$,

$$\varphi_n(\eta) \equiv P\left(\bigcup_{k=1}^n \left\{ \Gamma_j^{-1/\alpha} \frac{|f_k(U_j^{(n)})|}{\max_{1 \le i \le n} |f_i(U_j^{(n)})|} > \eta \text{ for at least 2 different } j = 1, 2, \dots \right\} \right)$$
$$\to 0 \tag{5.12}$$

for every $\eta > 0$. To see this, observe first that $n^{-1/2\alpha} b_n \to \infty$ as $n \to \infty$. By virtue of Remark 4.2 in Samorodnitsky (2004),

$$P\left(\bigcup_{k=1}^n \left\{ \frac{|f_k(U_j^{(n)})|}{\max_{1 \le i \le n} |f_i(U_j^{(n)})|} > \eta, \;\; j = 1, 2 \right\} \right) \to 0$$

holds for every $\eta > 0$ and, subsequently, the same argument as in p. 1453 of Samorodnitsky (2004) leads us to (5.12).

Having fixed $\lambda_1, \dots, \lambda_d > 0$, it will be proved that for every $\delta > 0$,

$$P\big(b_n^{-1} \max_{1 \le k \le \lfloor nt_i \rfloor} |X_k| > \lambda_i, \;\; i = 1, \dots, d\big)$$
$$\le P\left(C_\alpha^{1/\alpha} \bigvee_{j=1}^\infty \Gamma_j^{-1/\alpha} \frac{\max_{1 \le k \le \lfloor nt_i \rfloor} |f_k(U_j^{(n)})|}{\max_{1 \le k \le n} |f_k(U_j^{(n)})|} > \lambda_i(1-\delta), \;\; i = 1, \dots, d \right) + o(1) \tag{5.13}$$

and that

$$P\big(b_n^{-1} \max_{1 \le k \le \lfloor nt_i \rfloor} |X_k| > \lambda_i, \;\; i = 1, \dots, d\big)$$
$$\ge P\left(C_\alpha^{1/\alpha} \bigvee_{j=1}^\infty \Gamma_j^{-1/\alpha} \frac{\max_{1 \le k \le \lfloor nt_i \rfloor} |f_k(U_j^{(n)})|}{\max_{1 \le k \le n} |f_k(U_j^{(n)})|} > \lambda_i(1+\delta), \;\; i = 1, \dots, d \right) + o(1) \,. \tag{5.14}$$

Since the argument for (5.13) and the argument for (5.14) are very similar, we only prove (5.13). Let $K \in \mathbb{N}$ and $0 < \epsilon < 1$ be constants so that

$$K + 1 > \frac{4}{\alpha} \quad \text{and} \quad \delta - \epsilon K > 0\,.$$

Then, we proceed as follows:

$$\begin{aligned}
&P\big(b_n^{-1} \max_{1\le k\le \lfloor nt_i\rfloor} |X_k| > \lambda_i, \;\; i = 1, \dots, d\big) \\
&\le P\left(C_\alpha^{1/\alpha} \bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \frac{\max_{1\le k\le \lfloor nt_i\rfloor} |f_k(U_j^{(n)})|}{\max_{1\le k\le n} |f_k(U_j^{(n)})|} > \lambda_i(1-\delta), \;\; i = 1, \dots, d \right) \\
&\qquad + \varphi_n\big(C_\alpha^{-1/\alpha} \epsilon \min_{1\le i\le d} \lambda_i\big) \\
&\quad + \sum_{i=1}^{d} P\left(C_\alpha^{1/\alpha} \max_{1\le k\le \lfloor nt_i\rfloor} \left| \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \frac{f_k(U_j^n)}{\max_{1\le i\le n} |f_i(U_j^{(n)})|} \right| > \lambda_i\,, \right. \\
&\quad C_\alpha^{1/\alpha} \bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \frac{\max_{1\le k\le \lfloor nt_i\rfloor} |f_k(U_j^{(n)})|}{\max_{1\le k\le n} |f_k(U_j^{(n)})|} \le \lambda_i(1-\delta)\,, \;\; \text{and for each } m = 1, \dots, n, \\
&\quad \left. C_\alpha^{1/\alpha} \Gamma_j^{-1/\alpha} \frac{|f_m(U_j^{(n)})|}{\max_{1\le i\le n} |f_i(U_j^{(n)})|} > \epsilon \min_{1\le i\le d} \lambda_i \;\; \text{for at most one } j = 1, 2, \dots \right).
\end{aligned}$$

By (5.12), it is enough to show that for all $\lambda > 0$ and $0 \le t \le 1$,

$$\begin{aligned}
&P\left(C_\alpha^{1/\alpha} \max_{1\le k\le \lfloor nt\rfloor} \left| \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \frac{f_k(U_j^n)}{\max_{1\le i\le n} |f_i(U_j^{(n)})|} \right| > \lambda\,, \right. \qquad (5.15) \\
&C_\alpha^{1/\alpha} \bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \frac{\max_{1\le k\le \lfloor nt\rfloor} |f_k(U_j^{(n)})|}{\max_{1\le k\le n} |f_k(U_j^{(n)})|} \le \lambda(1-\delta)\,, \;\; \text{and for each } m = 1, \dots, n, \\
&\left. C_\alpha^{1/\alpha} \Gamma_j^{-1/\alpha} \frac{|f_m(U_j^{(n)})|}{\max_{1\le i\le n} |f_i(U_j^{(n)})|} > \epsilon\lambda \;\; \text{for at most one } j = 1, 2, \dots \right) \to 0\,.
\end{aligned}$$

Let

$$\|f\|_\alpha = \left(\int_E |f(x)|^\alpha \mu(dx) \right)^{1/\alpha}.$$

For every $k = 1, 2, \dots, n$, the Poisson random measure represented by the points

$$\big(\epsilon_j \Gamma_j^{-1/\alpha} f_k(U_j^{(n)}) \big(\max_{1\le i\le n} |f_i(U_j^{(n)})|\big)^{-1}, \; j = 1, 2, \dots\big)$$

has the same mean measure as that represented by the points

$$\big(\epsilon_j \Gamma_j^{-1/\alpha} \|f\|_\alpha b_n^{-1}, \; j = 1, 2, \dots\big)\,.$$

Namely, these two Poisson random measures distributionally coincide.

From this fact, the probability in the left hand side of (5.15) is bounded by

$$\sum_{k=1}^{\lfloor nt \rfloor} P\Bigg(C_\alpha^{1/\alpha} \left| \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \frac{f_k(U_j^{(n)})}{\max_{1\le i\le n} |f_i(U_j^{(n)})|} \right| > \lambda \,,$$

$$C_\alpha^{1/\alpha} \bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \frac{f_k(U_j^{(n)})}{\max_{1\le i\le n} |f_i(U_j^{(n)})|} \le \lambda(1-\delta)\,,$$

$$C_\alpha^{1/\alpha} \Gamma_j^{-1/\alpha} \frac{|f_k(U_j^{(n)})|}{\max_{1\le i\le n} |f_i(U_j^{(n)})|} > \epsilon\lambda \ \text{ for at most one } j = 1, 2, \dots \Bigg)$$

$$= \lfloor nt \rfloor P\Bigg(C_\alpha^{1/\alpha} \left| \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \right| > \lambda \|f\|_\alpha^{-1} b_n \,, \ \ C_\alpha^{1/\alpha} \bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \le \lambda(1-\delta) \|f\|_\alpha^{-1} b_n \,,$$

$$C_\alpha^{1/\alpha} \Gamma_j^{-1/\alpha} > \epsilon\lambda \|f\|_\alpha^{-1} b_n \ \text{ for at most one } j = 1, 2, \dots \Bigg)$$

$$\le nP\left(C_\alpha^{1/\alpha} \left| \sum_{j=K+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \right| > (\delta - \epsilon K)\lambda \|f\|_\alpha^{-1} b_n \right)$$

$$\le \frac{n\|f\|_\alpha^4 C_\alpha^{4/\alpha}}{(\delta-\epsilon K)^4 \lambda^4 b_n^4}\, E\left| \sum_{j=K+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \right|^4 .$$

Due to the constraint $K + 1 > 4/\alpha$,

$$E\left| \sum_{j=K+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \right|^4 < \infty;$$

see Samorodnitsky (2004) for a detailed proof. Since $n/b_n^4 \to 0$ as $n \to \infty$, (5.15) is obtained.

A constant $\delta > 0$ in (5.13) and (5.14) is arbitrarily chosen. Hence, the proof of finite-dimensional weak convergence will be finished if we can show convergence in law of the form

$$\left(\bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \frac{\max_{1\le k\le \lfloor nt_i \rfloor} |f_k(U_j^{(n)})|}{\max_{1\le k\le n} |f_k(U_j^{(n)})|},\ i = 1, \dots, d \right) \Rightarrow (Z_{\alpha,\beta}(t_i),\ i = 1, \dots, d) \quad \text{in } \mathbb{R}_+^d \,. \tag{5.16}$$

Let $\sum_{j=1}^{\infty} \delta_{(\Gamma_j, U_j^{(n)})}$ denote the Poisson random measure with mean measure $\lambda \times \eta_n$ (δ represents Dirac measure and λ is the one-dimensional Lebesgue measure on $\mathbb{R}_+$), and

$S_n : \mathbb{R}_+ \times E \to \mathbb{R}_+$ is defined by

$$S_n(r,x) = r^{-1/\alpha}\big(M_n(f)(x)\big)^{-1}\big(M_{\lfloor nt_1\rfloor}(f)(x),\dots,M_{\lfloor nt_d\rfloor}(f)(x)\big)\,,\;\; r>0,\; x\in E\,.$$

Then, for $\lambda_1,\dots,\lambda_d>0$,

$$P\left(\bigvee_{j=1}^{\infty}\Gamma_j^{-1/\alpha}\frac{\max_{1\le k\le \lfloor nt_i\rfloor}|f_k(U_j^{(n)})|}{\max_{1\le k\le n}|f_k(U_j^{(n)})|}\le \lambda_i\,,\;\; i=1,\dots,d\right)$$

$$= P\left(\sum_{j=1}^{\infty}\delta_{S_n(\Gamma_j,U_j^{(n)})}\Big((0,\lambda_1]\times\cdots\times(0,\lambda_d]\Big)^c=0\right)$$

$$= \exp\left\{-(\lambda\times\eta_n)\circ S_n^{-1}\Big((0,\lambda_1]\times\cdots\times(0,\lambda_d]\Big)^c\right\}$$

$$= \exp\left\{-(\lambda\times\eta_n)\left\{(r,x):\bigvee_{j=1}^{d}\lambda_j^{-\alpha}\frac{\big(M_{\lfloor nt_j\rfloor}(f)(x)\big)^{\alpha}}{\big(M_n(f)(x)\big)^{\alpha}}>r\right\}\right\}$$

$$= \exp\left\{-b_n^{-\alpha}\int_E\bigvee_{j=1}^{d}\lambda_j^{-\alpha}M_{\lfloor nt_j\rfloor}(f)^{\alpha}d\mu\right\}\,.$$

It follows from (5.10) that

$$\frac{1}{b_n^{\alpha}}\int_E\bigvee_{j=1}^{d}\lambda_j^{-\alpha}M_{\lfloor nt_j\rfloor}(f)^{\alpha}d\mu\sim\left(\int_A M_\infty(f)^{\alpha}d\eta\right)^{-1}\int_E\bigvee_{j=1}^{d}\lambda_j^{-\alpha}M_{\lfloor nt_j\rfloor}(f)^{\alpha}d\mu_n\,. \tag{5.17}$$

Then, Proposition 2.2.7 yields

$$\mu_n\circ\left(\bigvee_{j=1}^{d}\lambda_j^{-1}M_{\lfloor nt_j\rfloor}(f)\right)^{-1}\Rightarrow\big(\eta\times P'\big)\circ\left(M_\infty(f)\bigvee_{j=1}^{d}\lambda_j^{-1}\mathbf{1}_{\{V_\beta\le t_j\}}\right)^{-1}\quad\text{in }\mathbb{R}_+\,,$$

where V_β is a random variable defined on a probability space $(\Omega',\mathcal{F}',P')$ with $P'\big(V_\beta\le x\big)=x^\beta$, $0<x\le 1$. Thus, from (5.17) and the uniform integrability of $\big(M_n(f)^\alpha,\, n\ge 1\big)$ with respect to μ_n,

$$\frac{1}{b_n^{\alpha}}\int_E\bigvee_{j=1}^{d}\lambda_j^{-\alpha}M_{\lfloor nt_j\rfloor}(f)^{\alpha}d\mu\to\int_{\Omega'}\bigvee_{j=1}^{d}\lambda_j^{-\alpha}\mathbf{1}_{\{V_\beta\le t_j\}}dP'$$

$$=\sum_{i=1}^{d}\int_{t_{i-1}}^{t_i}\bigvee_{j=i}^{d}\lambda_j^{-\alpha}\beta x^{\beta-1}dx=\sum_{i=1}^{d}(t_i^\beta-t_{i-1}^\beta)\left(\bigwedge_{j=i}^{d}\lambda_j\right)^{-\alpha}.$$

Now we conclude

$$P\left(\bigvee_{j=1}^{\infty}\Gamma_j^{-1/\alpha}\frac{\max_{1\le k\le \lfloor nt_i\rfloor}|f_k(U_j^{(n)})|}{\max_{1\le k\le n}|f_k(U_j^{(n)})|}\le \lambda_i\,,\;\; i=1,\dots,d\right)$$

$$\to \exp\left\{-\sum_{i=1}^{d}(t_i^\beta - t_{i-1}^\beta)\left(\bigwedge_{j=i}^{d}\lambda_j\right)^{-\alpha}\right\}.$$

On the other hand, from the definition and the properties of $\big(Z_{\alpha,\beta}(t),\, t\geq 0\big)$, it is easy to verify that

$$P\big(Z_{\alpha,\beta}(t_i)\leq \lambda_i,\ \ i=1,\dots,d\big) = \exp\left\{-\sum_{i=1}^{d}(t_i^\beta - t_{i-1}^\beta)\left(\bigwedge_{j=i}^{d}\lambda_j\right)^{-\alpha}\right\}.$$

Therefore, (5.16) has been completed.

In the case of $f = \mathbf{1}_A$, we can establish weak convergence in the space $D[0,\infty)$ endowed with the J_1-topology. By virtue of the regular variation of (b_n), we only need to prove the weak convergence in the space $D[0,1]$. Truncating series representation (5.11) by the first K terms, it will be shown that

$$C_\alpha^{1/\alpha}\max_{1\leq k\leq \lfloor nt\rfloor}\left|\sum_{j=1}^{K}\epsilon_j\Gamma_j^{-1/\alpha}\mathbf{1}_A\circ T^k(U_j^{(n)})\right| \Rightarrow C_\alpha^{1/\alpha}\bigvee_{j=1}^{K}\Gamma_j^{-1/\alpha}\mathbf{1}_{\{V_j\leq t\}}\ \text{ in } D[0,1]\,, \tag{5.18}$$

where (V_j) are i.i.d. random variables with $P(V_j\leq x) = x^\beta$, $0<x\leq 1$. Assume for simplicity that (V_j) lives on the same probability space as the random variables on the left hand side.

For the proof of the finite-dimensional weak convergence, fix $0=t_0<t_1<\cdots<t_d\leq 1$, $d\geq 1$. By the same (or even easier) argument as the one leading to (5.13) and (5.14), it suffices to show the following:

$$\begin{aligned}&\left(\bigvee_{j=1}^{K}\Gamma_j^{-1/\alpha}\max_{1\leq k\leq\lfloor nt_i\rfloor}\mathbf{1}_A\circ T^k(U_j^{(n)}),\ i=1,\dots,d\right) \\ &\Rightarrow \left(\bigvee_{j=1}^{K}\Gamma_j^{-1/\alpha}\mathbf{1}_{\{V_j\leq t_i\}},\ i=1,\dots,d\right)\ \text{ in } \mathbb{R}_+^d\,.\end{aligned} \tag{5.19}$$

First, we will prove

$$\left(\max_{1\leq k\leq\lfloor nt_i\rfloor}\mathbf{1}_A\circ T^k(U_j^{(n)}),\ i=1,\dots,d\right)\Rightarrow\big(\mathbf{1}_{\{V_j\leq t_i\}},\ i=1,\dots,d\big)\ \text{ in } \mathbb{R}_+^d$$

for every $j=1,\dots,K$. To see this, we fix $\theta_1,\dots,\theta_d>0$ and $j\in\{1,\dots,K\}$. Then

$$P\left(\bigvee_{i=1}^{d}\theta_i\max_{1\leq k\leq\lfloor nt_i\rfloor}\mathbf{1}_A\circ T^k(U_j^{(n)})>\lambda\right)$$

$$= \frac{1}{b_n^\alpha} \int_E \mathbf{1} \left(\bigvee_{i=1}^d \theta_i M_{\lfloor nt_i \rfloor}(\mathbf{1}_A) > \lambda \right) M_n(\mathbf{1}_A)^\alpha d\mu = \mu_n \left(\bigvee_{i=1}^d \theta_i M_{\lfloor nt_i \rfloor}(\mathbf{1}_A) > \lambda \right) .$$

Observing that $M_\infty(\mathbf{1}_A) = 1$ a.e. on A, Proposition 2.2.7 gives

$$\mu_n \left(\bigvee_{i=1}^d \theta_i M_{\lfloor nt_i \rfloor}(\mathbf{1}_A) > \lambda \right) \to P' \left(\bigvee_{i=1}^d \theta_i \mathbf{1}_{\{V_\beta \le t_i\}} > \lambda \right) = P \left(\bigvee_{i=1}^d \theta_i \mathbf{1}_{\{V_j \le t_i\}} > \lambda \right) .$$

Since $(U_j^{(n)})$ and (Γ_j) are independent, we obtain (5.19).

Let

$$Y_j^{(k,n)} = \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}_A \circ T^k(U_j^{(n)}) \,,$$

and we will show the tightness of $\left(\max_{1 \le k \le \lfloor nt \rfloor} | \sum_{j=1}^K Y_j^{(k,n)} | \,,\, 0 \le t \le 1 \right)$ in the space $D[0,1]$. To this aim, let

$$B_n^{(K)} = \bigcap_{m=1}^n \left\{ U_j^{(n)} \in T^{-m} A \ \text{ for at most one } j \in \{1, \dots, K\} \right\} .$$

Then $B_n^{(K)}$ is a set of asymptotic probability 1 when $n \to \infty$;

$$P\Big(\big(B_n^{(K)} \big)^c \Big) = P \left(\bigcup_{k=1}^n \left\{ U_j^{(n)} \in T^{-k} A \ \text{ for at least 2 different } j \in \{1, \dots, K\} \right\} \right)$$

$$\le K^2 P \left(\bigcup_{k=1}^n \left\{ U_1^{(n)} \in T^{-k} A,\, U_2^{(n)} \in T^{-k} A \right\} \right)$$

$$\le K^2 \sum_{k=1}^n P\big(U_1^{(n)} \in T^{-k} A \big)^2 \le K^2 \mu(A)^2 \frac{n}{b_n^{2\alpha}} \to 0 \,.$$

Therefore, we only have to prove the tightness of $\left(\max_{1 \le k \le \lfloor nt \rfloor} | \sum_{j=1}^K Y_j^{(k,n)} | \mathbf{1}_{B_n^{(K)}} \,,\, 0 \le t \le 1 \right)$. The sufficient condition we shall check is given by Theorem 13.5 of Billingsley (1999): there exist constants $B > 0$ and $\gamma > 1$ such that

$$P\Bigg(\max_{1 \le k \le \lfloor nt \rfloor} \left| \sum_{j=1}^K Y_j^{(k,n)} \right| \mathbf{1}_{B_n^{(K)}} - \max_{1 \le k \le \lfloor ns \rfloor} \left| \sum_{j=1}^K Y_j^{(k,n)} \right| \mathbf{1}_{B_n^{(K)}} \ge \lambda \,, \tag{5.20}$$

$$\max_{1 \le k \le \lfloor ns \rfloor} \left| \sum_{j=1}^K Y_j^{(k,n)} \right| \mathbf{1}_{B_n^{(K)}} - \max_{1 \le k \le \lfloor nr \rfloor} \left| \sum_{j=1}^K Y_j^{(k,n)} \right| \mathbf{1}_{B_n^{(K)}} \ge \lambda \Bigg) \le B(t-r)^\gamma$$

for all $0 \le r \le s \le t \le 1$, $n \ge 1$ and $\lambda > 0$. If $t - r < 1/n$, then the probability in the left hand side vanishes; therefore, we may assume $t - r \ge 1/n$.

Let $D_{ik}^{(n)} = \{U_i^{(n)} \in T^{-k}A\}$. To estimate the probability in (5.20),

$$\left\{ \max_{1\le k\le \lfloor nt\rfloor} \left| \sum_{j=1}^{K} Y_j^{(k,n)} \right| \mathbf{1}_{B_n^{(K)}} - \max_{1\le k\le \lfloor ns\rfloor} \left| \sum_{j=1}^{K} Y_j^{(k,n)} \right| \mathbf{1}_{B_n^{(K)}} \ge \lambda \right\}$$

$$\subseteq \left\{ \max_{\lfloor ns\rfloor+1\le k\le \lfloor nt\rfloor} \left| \sum_{j=1}^{K} Y_j^{(k,n)} \right| > \max_{1\le k\le \lfloor ns\rfloor} \left| \sum_{j=1}^{K} Y_j^{(k,n)} \right| \right\} \cap B_n^{(K)}$$

$$= \left\{ \bigvee_{j=1}^{K} \Gamma_j^{-1/\alpha} \max_{\lfloor ns\rfloor+1\le k\le \lfloor nt\rfloor} \mathbf{1}_A \circ T^k(U_j^{(n)}) > \bigvee_{j=1}^{K} \Gamma_j^{-1/\alpha} \max_{1\le k\le \lfloor ns\rfloor} \mathbf{1}_A \circ T^k(U_j^{(n)}) \right\} \cap B_n^{(K)}$$

$$= \bigcup_{p=1}^{K} \Bigg(\left\{ \Gamma_p^{-1/\alpha} > \bigvee_{j=1}^{K} \Gamma_j^{-1/\alpha} \max_{1\le k\le \lfloor ns\rfloor} \mathbf{1}_A \circ T^k(U_j^{(n)}) \right\} \cap B_n^{(K)}$$

$$\cap \bigcap_{i=1}^{p-1} \bigcap_{k=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} (D_{ik}^{(n)})^c \cap \bigcup_{k=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} D_{pk}^{(n)} \Bigg)$$

$$= \bigcup_{p=1}^{K} \Bigg(\left\{ \Gamma_p^{-1/\alpha} > \bigvee_{j=1}^{K} \Gamma_j^{-1/\alpha} \max_{1\le k\le \lfloor ns\rfloor} \mathbf{1}_A \circ T^k(U_j^{(n)}) \right\} \cap B_n^{(K)}$$

$$\cap \bigcap_{i=1}^{p-1} \bigcap_{k=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} (D_{ik}^{(n)})^c \cap \bigcup_{k=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} D_{pk}^{(n)} \cap \bigcap_{k=1}^{\lfloor ns\rfloor} (D_{pk}^{(n)})^c \Bigg).$$

The last relation follows from

$$B_n^{(K)} \cap \bigcup_{k=1}^{\lfloor ns\rfloor} D_{pk}^{(n)} \subseteq \left\{ \bigvee_{j=1}^{K} \Gamma_j^{-1/\alpha} \max_{1\le k\le \lfloor ns\rfloor} \mathbf{1}_A \circ T^k(U_j^{(n)}) \ge \Gamma_p^{-1/\alpha} \right\}.$$

Therefore, it turns out that the left hand side of (5.20) is bounded by

$$P\left(\bigcup_{i=1}^{K} \left(\bigcap_{k=1}^{\lfloor ns\rfloor} (D_{ik}^{(n)})^c \cap \bigcup_{k=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} D_{ik}^{(n)} \right) \cap \bigcup_{j=1}^{K} \left(\bigcap_{k=1}^{\lfloor nr\rfloor} (D_{jk}^{(n)})^c \cap \bigcup_{k=\lfloor nr\rfloor+1}^{\lfloor ns\rfloor} D_{jk}^{(n)} \right) \right)$$

$$= P\left(\bigcup_{i=1}^{K} \bigcup_{j=1,\, i\ne j}^{K} \left(\bigcap_{k=1}^{\lfloor ns\rfloor} (D_{ik}^{(n)})^c \cap \bigcup_{k=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} D_{ik}^{(n)} \cap \bigcap_{k=1}^{\lfloor nr\rfloor} (D_{jk}^{(n)})^c \cap \bigcup_{k=\lfloor nr\rfloor+1}^{\lfloor ns\rfloor} D_{jk}^{(n)} \right) \right)$$

$$\le K^2 P\left(\bigcup_{k=\lfloor nr\rfloor+1}^{\lfloor nt\rfloor} D_{1k}^{(n)} \right)^2 = K^2 \left(\frac{\mu\big(\varphi \le \lfloor nt\rfloor - \lfloor nr\rfloor\big)}{\mu(\varphi \le n)} \right)^2.$$

Fix $\eta > 0$ with $2(\beta - \eta) > 1$. By the constraint $t - r \ge 1/n$ and the regular variation of $\mu(\varphi \le \cdot)$, there is a constant $C > 0$ such that

$$\frac{\mu\big(\varphi \le \lfloor nt\rfloor - \lfloor nr\rfloor\big)}{\mu(\varphi \le n)} \le \frac{\mu\big(\varphi \le \lfloor 2n(t-r)\rfloor\big)}{\mu(\varphi \le n)}$$

$$\le C\left(\frac{\lfloor 2n(t-r)\rfloor}{n}\right)^{\beta-\eta} \le 2^{\beta-\eta}C(t-r)^{\beta-\eta}\,.$$

Taking $B = 4^{\beta-\eta}K^2C^2$ and $\gamma = 2(\beta-\eta)$, the required condition (5.20) has been satisfied.

Next, we note that in the space $D[0,1]$,

$$C_\alpha^{1/\alpha}\bigvee_{j=1}^{K}\Gamma_j^{-1/\alpha}\mathbf{1}_{\{V_j\le t\}} \to C_\alpha^{1/\alpha}\bigvee_{j=1}^{\infty}\Gamma_j^{-1/\alpha}\mathbf{1}_{\{V_j\le t\}} \quad \text{as } K\to\infty \quad \text{a.s.} \tag{5.21}$$

This is because

$$\sup_{0\le t\le 1}\left(\bigvee_{j=1}^{\infty}\Gamma_j^{-1/\alpha}\mathbf{1}_{\{V_j\le t\}} - \bigvee_{j=1}^{K}\Gamma_j^{-1/\alpha}\mathbf{1}_{\{V_j\le t\}}\right)$$

$$= \sup_{0\le t\le 1}\bigvee_{j=K+1}^{\infty}\Gamma_j^{-1/\alpha}\mathbf{1}\left(V_j\le t,\ V_i>t,\ i=1,\ldots,K\right) \le \Gamma_{K+1}^{-1/\alpha} \to 0$$

as $K\to\infty$ a.s..

Now, we have checked two types of convergence (5.18) and (5.21). According to Theorem 3.2 in Billingsley (1999), in order to finish the proof, it remains to show that

$$\lim_{K\to\infty}\limsup_{n\to\infty} P\left(\max_{1\le k\le n}\left|\sum_{j=K+1}^{\infty}\epsilon_j\Gamma_j^{-1/\alpha}\mathbf{1}_A\circ T^k(U_j^{(n)})\right| > \epsilon\right) = 0$$

for every $\epsilon>0$. We may write

$$P\left(\max_{1\le k\le n}\left|\sum_{j=K+1}^{\infty}\epsilon_j\Gamma_j^{-1/\alpha}\mathbf{1}_A\circ T^k(U_j^{(n)})\right| > \epsilon\right)$$

$$= \int_0^\infty e^{-x}\frac{x^K}{K!}P\left(\max_{1\le k\le n}\left|\sum_{j=1}^{\infty}\epsilon_j\big(\Gamma_j+x\big)^{-1/\alpha}\mathbf{1}_A\circ T^k(U_j^{(n)})\right| > \epsilon\right)dx$$

$$\le \int_0^{(\epsilon/2)^{-\alpha}} e^{-x}\frac{x^K}{K!}dx + \int_{(\epsilon/2)^{-\alpha}}^\infty e^{-x}\frac{x^K}{K!}P\left(\max_{1\le k\le n}\left|\sum_{j=1}^{\infty}\epsilon_j\big(\Gamma_j+x\big)^{-1/\alpha}\mathbf{1}_A\circ T^k(U_j^{(n)})\right| > \epsilon\right)dx$$

Clearly, by Stirling's formula, the first term vanishes when $K\to\infty$. For the second term, we will show that for every $x\ge(\epsilon/2)^{-\alpha}$,

$$P\left(\max_{1\le k\le n}\left|\sum_{j=1}^{\infty}\epsilon_j\big(\Gamma_j+x\big)^{-1/\alpha}\mathbf{1}_A\circ T^k(U_j^{(n)})\right| > \epsilon\right) \to 0 \tag{5.22}$$

as $n\to\infty$.

To prove (5.22), by a direct consequence of (5.12),

$$P\left(\bigcup_{k=1}^{n}\left\{\big(\Gamma_j+x\big)^{-1/\alpha}\mathbf{1}_A\circ T^k(U_j^{(n)}) > \eta \ \text{ for at least 2 different } j=1,2,\ldots\right\}\right) \to 0$$

for every $\eta > 0$. Choose $L \in \mathbb{N}$ and $0 < \xi < 1/2$ so that

$$L + 1 > \frac{4}{\alpha} \quad \text{and} \quad \frac{1}{2} - \xi L > 0\,. \tag{5.23}$$

Since

$$P\left((\Gamma_1 + x)^{-1/\alpha} > \frac{\epsilon}{2} \right) = 0$$

for all $x \geq (\epsilon/2)^{-\alpha}$, we can write

$$\begin{aligned} & P\left(\max_{1\le k\le n} \left| \sum_{j=1}^{\infty} \epsilon_j \big(\Gamma_j + x\big)^{-1/\alpha} \mathbf{1}_A \circ T^k(U_j^{(n)}) \right| > \epsilon \right) \qquad (5.24) \\ & \leq P\Bigg(\max_{1\le k\le n} \left| \sum_{j=1}^{\infty} \epsilon_j \big(\Gamma_j + x\big)^{-1/\alpha} \mathbf{1}_A \circ T^k(U_j^{(n)}) \right| > \epsilon\,, \\ & \big(\Gamma_1 + x\big)^{-1/\alpha} \leq \frac{\epsilon}{2}\,, \ \text{ and for each } m = 1, \dots, n, \\ & \big(\Gamma_j + x\big)^{-1/\alpha} \mathbf{1}_A \circ T^m(U_j^{(n)}) > \xi\epsilon \ \text{ for at most one } j = 1, 2, \dots \Bigg) + o(1)\,. \end{aligned}$$

Notice that for every $k = 1, \dots, n$, the Poisson random measure represented by the points

$$\big(\epsilon_j \big(\Gamma_j + x\big)^{-1/\alpha} \mathbf{1}_A \circ T^k(U_j^{(n)}),\ j = 1, 2, \dots\big)$$

is distributionally equal to the Poisson random measure represented by the points

$$\big(\epsilon_j \big(b_n^\alpha \mu(A)^{-1} \Gamma_j + x\big)^{-1/\alpha},\ j = 1, 2, \dots\big)\,.$$

Now, the first term on the right hand side of (5.24) can be bounded by

$$\begin{aligned} & \sum_{k=1}^{n} P\Bigg(\left| \sum_{j=1}^{\infty} \epsilon_j \big(\Gamma_j + x\big)^{-1/\alpha} \mathbf{1}_A \circ T^k(U_j^{(n)}) \right| > \epsilon\,,\ \bigvee_{j=1}^{\infty} \big(\Gamma_j + x\big)^{-1/\alpha} \mathbf{1}_A \circ T^k(U_j^{(n)}) \leq \frac{\epsilon}{2}\,, \\ & \big(\Gamma_j + x\big)^{-1/\alpha} \mathbf{1}_A \circ T^k(U_j^{(n)}) > \xi\epsilon \ \text{ for at most one } j = 1, 2, \dots \Bigg) \\ & = nP\Bigg(\left| \sum_{j=1}^{\infty} \epsilon_j \big(b_n^\alpha \mu(A)^{-1} \Gamma_j + x\big)^{-1/\alpha} \right| > \epsilon\,,\ \bigvee_{j=1}^{\infty} \big(b_n^\alpha \mu(A)^{-1} \Gamma_j + x\big)^{-1/\alpha} \leq \frac{\epsilon}{2}\,, \\ & \big(b_n^\alpha \mu(A)^{-1} \Gamma_j + x\big)^{-1/\alpha} > \xi\epsilon \ \text{ for at most one } j = 1, 2, \dots \Bigg) \end{aligned}$$

$$\leq nP\left(\left|\sum_{j=L+1}^{\infty}\epsilon_j\big(b_n^\alpha\mu(A)^{-1}\Gamma_j+x\big)^{-1/\alpha}\right|>\left(\frac{1}{2}-\xi L\right)\epsilon\right).$$

By the contraction inequality for Rademacher series (see e.g. Proposition 1.2.1 of Kwapień and Woyczyński (1992)),

$$nP\left(\left|\sum_{j=L+1}^{\infty}\epsilon_j\big(b_n^\alpha\mu(A)^{-1}\Gamma_j+x\big)^{-1/\alpha}\right|>\left(\frac{1}{2}-\xi L\right)\epsilon\right)$$

$$\leq 2nP\left(\left|\sum_{j=L+1}^{\infty}\epsilon_j\Gamma_j^{-1/\alpha}\right|>\left(\frac{1}{2}-\xi L\right)\epsilon\mu(A)^{-1/\alpha}b_n\right).$$

Due to the constraints of the constants $L\in\mathbb{N}$ and $0<\xi<1/2$ given in (5.23),

$$2nP\left(\left|\sum_{j=L+1}^{\infty}\epsilon_j\Gamma_j^{-1/\alpha}\right|>\left(\frac{1}{2}-\xi L\right)\epsilon\mu(A)^{-1/\alpha}b_n\right)$$

$$\leq\frac{2n\mu(A)^{4/\alpha}}{(2^{-1}-\xi L)^4\epsilon^4b_n^4}\,E\left|\sum_{j=L+1}^{\infty}\epsilon_j\Gamma_j^{-1/\alpha}\right|^4\to 0$$

as $n\to\infty$ and hence (5.22) is complete. □

5.4 Examples

Example 5.4.1. We consider the same setup as Example 3.3.5, that is, let $(x_n,\ n\in\mathbb{N})$ be an irreducible null recurrent Markov chain with state space $\mathbb{Z}$ and transition matrix $P=(p_{ij})$. Let $\{\pi_j, j\in\mathbb{Z}\}$ be the unique invariant measure of the Markov chain that satisfies $\pi_0=1$. We define a σ-finite measure on $(E,\mathcal{E})=\big(\mathbb{Z}^{\mathbb{N}},\mathcal{B}(\mathbb{Z}^{\mathbb{N}})\big)$ by

$$\mu(\cdot)=\sum_{i\in\mathbb{Z}}\pi_iP_i(\cdot),$$

with the usual notation of $P_i(\cdot)$ being the probability law of the Markov chain starting in state $i\in\mathbb{Z}$. Let $T:\mathbb{Z}^{\mathbb{N}}\to\mathbb{Z}^{\mathbb{N}}$ be the left shift map $T(x_0,x_1,\dots)=(x_1,x_2,\dots)$ for $\{x_k, k=0,1,\dots\}\in\mathbb{Z}^{\mathbb{N}}$. Obviously, T preserves the measure μ. Since the Markov chain is irreducible and null recurrent, the flow $\{T^n\}$ is conservative and ergodic; see Harris and Robbins (1953).

Consider the set $A = \{x \in \mathbb{Z}^{\mathbb{N}} : x_0 = 0\}$ and the corresponding first entrance time $\varphi(x) = \min\{n \geq 1 : x_n = 0\}$, $x \in \mathbb{Z}^{\mathbb{N}}$. Assume that

$$\sum_{k=1}^{n} P_0(\varphi \geq k) \in RV_\beta$$

for some $\beta \in (1/2, 1)$. As shown in Example 3.3.5, A turns out to be a Darling-Kac set and hence, T is pointwise dual ergodic with normalizing sequence (a_n) that is regularly varying with index $1-\beta$. Furthermore, condition (5.4) has been seen to hold in Example 3.3.5.

We, thus, conclude that Theorem 5.3.1 applies if we choose any $L^\alpha(\mu)$-function f supported by A and satisfying condition (5.5).

Example 5.4.2. We consider the basic AFN-system introduced in Example 3.3.6. The setup of the basic AFN-system here is the same as Example 3.3.6 except condition (3.23). Indeed, we assume that for every $Z \in \zeta$, there is $1/2 < \beta_Z < 1$ such that

$$Tx = x + a_Z|x - x_Z|^{(1-\beta_Z)^{-1}+1} + o\big(|x - x_Z|^{(1-\beta_Z)^{-1}+1}\big) \quad \text{as } x \to x_Z \text{ in } Z,$$

for some $a_Z \neq 0$. Let $\beta = \max_{Z\in\zeta} \beta_Z \in (1/2, 1)$. Then the normalizing sequence (a_n) for pointwise dual ergodicity is regularly varying with exponent $1 - \beta$. We then conclude that Theorem 5.3.1 applies if any $L^\alpha(\mu)$-function f is supported by a Darling-Kac set A and satisfies (5.5).

CHAPTER 6

TAIL MEASURES

6.1 Tail Measures and Their Properties

Let T be an arbitrary (possibly infinite) index set and let $(X_t,\, t \in T)$ be a stochastic process or a random field, e.g., $T = \mathbb{Z}$ for a univariate time series, $T = \mathbb{Z} \times \{1, \dots, k\}$ for a k-dimensional time series, and $T = \mathbb{R}^d$ for a random field. It is assumed throughout this chapter that for any parameter space T, $(X_t,\, t \in T)$ has regularly varying tails. More precisely, we suppose that there exists a function $H : (0, \infty) \to (0, \infty)$ growing to infinity such that for all $t_1, \dots, t_k \in T$, $k \geq 1$, there is a Radon measure $\mu_{t_1 \dots t_k}$ on $\overline{\mathbb{R}}^k \setminus \{\mathbf{0}\}$ with $\mu(\overline{\mathbb{R}}^k \setminus \mathbb{R}^k) = 0$, such that as $u \to \infty$,

$$H(u) P\Big(\big(X_{t_1}, \dots, X_{t_k} \big) \in u \,\cdot \Big) \xrightarrow{v} \mu_{t_1 \dots t_k}(\cdot) \tag{6.1}$$

vaguely in $\overline{\mathbb{R}}^k \setminus \{\mathbf{0}\}$. We assume that for at least one $t_0 \in T$, μ_{t_0} is a nonzero measure. With this assumption, the standard argument regarding regular variation (e.g., Resnick (2007)) shows that $H(x)$ is regularly varying with index $-\alpha$ for some $\alpha > 0$. Furthermore, the Radon measure $\mu_{t_1 \dots t_k}$, $t_1, \dots, t_k \in T$ satisfies the homogeneity property: $\mu_{t_1 \dots t_k}(sA) = s^{-\alpha} \mu_{t_1 \dots t_k}(A)$ for all Borel sets $A \subseteq \overline{\mathbb{R}}^k \setminus \{\mathbf{0}\}$ and $s > 0$. The existence of the homogeneity exponent α is often emphasized by saying that $(X_t,\, t \in T)$ has regularly varying tails with index α.

Each Radon measure $\mu_{t_1 \dots t_k}$ can be seen to contain information regarding the high-level dependence of $(X_{t_1}, \dots, X_{t_k})$. However, merely observing each $\mu_{t_1 \dots t_k}$ will be insufficient if one hopes to comprehensively capture the extremal behavior of a stochastic process or a random field as a whole. This is particularly true for the relation between the probability laws of finite-dimensional random vectors and the probability law of a stochastic process. We may fail to keep track of the way a stochastic process evolves dynamically if we focus solely on a family of finite-dimensional random vectors. However, the Kolmogorov extension theorem allows one to clarify the connection between the finite-dimensional random vectors and the corresponding stochastic process. Indeed, given a family of probability laws of these vectors, this theorem guarantees the existence of the corresponding stochastic process. On the other hand, the construction of an infinite-dimensional object that unifies the family of finite-dimensional Radon measures in (6.1) is a nontrivial matter. This is because Radon measures in (6.1) blow up at the origin, whence $\mu_{t_1 \dots t_k}(\overline{\mathbb{R}}^k \setminus \{\mathbf{0}\}) = \infty$, and this disallows standard use of Kolmogorov extension theorem.

However, it is still possible to prove the existence of such an infinite-dimensional measure using a method suggested by Maruyama (1970). Let T be an arbitrary parameter space and let $(Y_t,\, t \in T)$ be an infinitely divisible process. That is, for all $k \geq 1$ and $t_1, \dots, t_k \in T$, $\big(Y_{t_1}, \dots, Y_{t_k}\big)$ forms an infinitely divisible random vector. Specifically, the law of $\big(Y_{t_1}, \dots, Y_{t_k}\big)$ is identified by a triplet $\big(\Sigma_F, \rho_F, \mathbf{b}_F\big)$, $F = \{t_1, \dots, t_k\}$, where Σ_F is the covariance matrix of the Gaussian part, $\mathbf{b}_F \in \mathbb{R}^F$, and ρ_F is the Lévy measure. Importantly, the system of Lévy measures $\{\rho_F : F \subseteq T, \text{finite}\}$ is consistent in the sense that for all finite index sets $F \subseteq G \subseteq T$,

$$\rho_F(B) = \rho_G\Big(p_{GF}^{-1}\big(B \setminus \{\mathbf{0}_F\}\big)\Big), \quad B \in \mathcal{B}(\mathbb{R}^F)\,, \tag{6.2}$$

where $p_{GF} : \mathbb{R}^G \to \mathbb{R}^F$ is the projection (represented by a $|F| \times |G|$ matrix), and $\mathbf{0}_F$ is the origin of $\mathbb{R}^F$. Furthermore, (6.2) implies that every Lévy measure has no mass at the origin, i.e., for every finite $F \subseteq T$,

$$\rho_F\big(\{\mathbf{0}_F\}\big) = 0\,. \tag{6.3}$$

Exploiting the structural properties (6.2) (and (6.3)) of finite-dimensional Lévy measures, Maruyama (1970) proves the existence of *a big triplet* $\big(\Sigma, \rho, \mathbf{b}\big)$ that characterizes the distribution of the *whole* process $\mathbf{Y} = (Y_t,\, t \in T)$. As a result, one can reconstruct each triplet $(\Sigma_F, \rho_F, \mathbf{b}_F)$ from $(\Sigma, \rho, \mathbf{b})$ by

$$\begin{cases} \Sigma_F = p_F \Sigma p_F^T\,, \\ \rho_F(B) = \rho\Big(p_F^{-1}\big(B \setminus \{\mathbf{0}\}\big)\Big), \quad B \in \mathcal{B}(\mathbb{R}^F)\,, \\ \mathbf{b}_F = p_F \mathbf{b}\,, \end{cases}$$

where $p_F : \mathbb{R}^T \to \mathbb{R}^F$ is the projection given by $p_F \mathbf{x} = \mathbf{x}|_F$, $\mathbf{x} \in \mathbb{R}^T$, and its adjoint p_F^T satisfies

$$\big(p_F^T z\big)_t = \begin{cases} z_t & t \in F, \\ 0 & t \in T \setminus F\,. \end{cases}$$

The situation for Radon measures defined by (6.1) is analogous to that for finite-dimensional Lévy measures of an infinitely divisible process. Indeed, in Theorem 6.1.1 below, slightly modified versions of the Radon measures in (6.1) will be shown to satisfy (6.2) and (6.3). Consequently, the resulting cylindrical measure turns out to be well-defined on the space $\mathbb{R}^T$.

Theorem 6.1.1. *Let $\mathbf{X} = (X_t,\ t \in T)$ be a stochastic process or a random field, assuming regularly varying tails in the sense of (6.1). Let $\mathcal{B}(\mathbb{R})^T = \prod_{t \in T} \mathcal{B}(\mathbb{R}_t)$, where $\mathbb{R}_t = \mathbb{R}$, be the cylindrical σ-field on $\mathbb{R}^T$. Then a cylindrical measure ν on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^T)$, satisfying (i) and (ii) below, uniquely exists. This measure is called a tail measure of* $\mathbf{X}$.

(i) : *For any finite index set $F = \{t_1, \ldots, t_k\} \subseteq T$,*

$$\mu_F(A) = \nu\big(p_F^{-1}(A)\big)$$

for every Borel set $A \subseteq \mathbb{R}^F \setminus \{\mathbf{0}_F\}$.

(ii) : *For every countable $T_1 \subseteq T$, there exists a countable set T_2, such that $T_1 \subseteq T_2 \subseteq T$ and*

$$\nu\Big(p_{T_1}^{-1}\big(\{\mathbf{0}_{T_1}\}\big)\Big) = \nu\Big(p_{T_1}^{-1}\big(\{\mathbf{0}_{T_1}\}\big) \setminus p_{T_2}^{-1}\big(\{\mathbf{0}_{T_2}\}\big)\Big)\,.$$

Remark 6.1.2. Condition (ii) in Theorem 6.1.1 indirectly tells us that a tail measure ν has no mass at the origin (note that if T is uncountable, then the statement $\nu\big(\{\mathbf{0}_T\}\big) = 0$ does not make sense, because $\{\mathbf{0}_T\}$ is not measurable in $\mathbb{R}^T$). As an evidence, we can show that if T is countable, condition (ii) is equivalent to $\nu\big(\{\mathbf{0}_T\}\big) = 0$. To see this, condition (ii) implies

$$\nu\big(\{\mathbf{0}_T\}\big) = \nu\Big(p_T^{-1}\big(\{\mathbf{0}_T\}\big)\Big) = \nu\Big(p_T^{-1}\big(\{\mathbf{0}_T\}\big) \setminus p_T^{-1}\big(\{\mathbf{0}_T\}\big)\Big) = \nu(\emptyset) = 0\,.$$

Conversely, if $\nu\big(\{\mathbf{0}_T\}\big) = 0$, then for every countable set $T_1 \subseteq T$,

$$\nu\Big(p_{T_1}^{-1}\big(\{\mathbf{0}_{T_1}\}\big) \setminus p_T^{-1}\big(\{\mathbf{0}_T\}\big)\Big) = \nu\Big(p_{T_1}^{-1}\big(\{\mathbf{0}_{T_1}\}\big)\Big) - \nu\big(\{\mathbf{0}_T\}\big) = \nu\Big(p_{T_1}^{-1}\big(\{\mathbf{0}_{T_1}\}\big)\Big)\,.$$

Remark 6.1.3. If there exists a countable set $T_0 \subseteq T$ such that $\nu\Big(p_{T_0}^{-1}\big(\{\mathbf{0}_{T_0}\}\big)\Big) = 0$, then condition (ii) follows. To show this, we take an arbitrary countable set $T_1 \subseteq T$. Define $T_2 = T_0 \cup T_1$, which is still countable. Since $\nu\Big(p_{T_2}^{-1}\big(\{\mathbf{0}_{T_2}\}\big)\Big) \leq \nu\Big(p_{T_0}^{-1}\big(\{\mathbf{0}_{T_0}\}\big)\Big) = 0$,

$$\nu\Big(p_{T_1}^{-1}\big(\{\mathbf{0}_{T_1}\}\big) \setminus p_{T_2}^{-1}\big(\{\mathbf{0}_{T_2}\}\big)\Big) = \nu\Big(p_{T_1}^{-1}\big(\{\mathbf{0}_{T_1}\}\big)\Big) - \nu\Big(p_{T_2}^{-1}\big(\{\mathbf{0}_{T_2}\}\big)\Big) = \nu\Big(p_{T_1}^{-1}\big(\{\mathbf{0}_{T_1}\}\big)\Big)\,.$$

As we will see in Proposition 6.1.4 below, if one can find such a countable set $T_0 \subseteq T$, then ν turns out to be σ-finite.

Proof. First, we identify every μ_F, with finite $F \subseteq T$, defined on $\overline{\mathbb{R}}^F \setminus \{\mathbf{0}_F\}$ by a measure ν_F on $\mathbb{R}^F$ as follows:

$$\begin{cases} \nu_F\big(\{\mathbf{0}_F\}\big) = 0\,, \\ \nu_F(A) = \mu_F(A) \;\; \text{for any Borel set } A \subseteq \mathbb{R}^F \setminus \{\mathbf{0}_F\}\,. \end{cases}$$

We claim that for any finite sets $F \subseteq G \subseteq T$,

$$\nu_F(B) = \nu_G\Big(p_{GF}^{-1}\big(B \setminus \{\mathbf{0}_F\}\big)\Big), \quad B \in \mathcal{B}(\mathbb{R}^F)\,. \tag{6.4}$$

We first show (6.4) for every $B \in \mathcal{B}(\mathbb{R}^F)$ with $\mathbf{0}_F \notin B$. Fix $G \subseteq T$ and prove this inductively with respect to $\dim(F) \in \{1, \ldots, \dim(G)\}$. Suppose $\dim(F) = 1$. Then, it suffices to show that

$$\nu_F\big((-\infty, -a] \cup [b, \infty)\big) = \nu_G\Big(p_{GF}^{-1}\big((-\infty, -a] \cup [b, \infty)\big)\Big)$$

for every $a > 0$, $b > 0$. We can assume without loss of generality that $(-\infty, -a] \cup [b, \infty)$ and $p_{GF}^{-1}\big((-\infty, -a] \cup [b, \infty)\big)$ both are continuity sets. Since $(-\infty, -a] \cup [b, \infty)$ is relatively compact on $\overline{\mathbb{R}}^F \setminus \{0\}$, the Portmanteau theorem for vague convergence (see e.g. Proposition 3.12 in Resnick (1987)) gives

$$\nu_F((-\infty,-a]\cup[b,\infty)) = \mu_F((-\infty,-a]\cup[b,\infty))$$

$$= \lim_{u\to\infty} H(u)P\big(u^{-1}X_F \in (-\infty,-a]\cup[b,\infty)\big)$$

$$= \lim_{u\to\infty} H(u)P\Big(u^{-1}\mathbf{X}_G \in p_{GF}^{-1}\big((-\infty,-a]\cup[b,\infty)\big)\Big).$$

Since $p_{GF}^{-1}\big((-\infty,-a]\cup[b,\infty)\big)$ is relatively compact on $\overline{\mathbb{R}}^G \setminus \{\mathbf{0}_G\}$, one more application of the Portmanteau theorem concludes

$$\lim_{u\to\infty} H(u)P\Big(u^{-1}\mathbf{X}_G \in p_{GF}^{-1}\big((-\infty,-a]\cup[b,\infty)\big)\Big)$$

$$= \mu_G\Big(p_{GF}^{-1}\big((-\infty,-a]\cup[b,\infty)\big)\Big) = \nu_G\Big(p_{GF}^{-1}\big((-\infty,-a]\cup[b,\infty)\big)\Big).$$

Next, suppose that (6.4) is true as long as $1 \le \dim(F) \le m < \dim(G)$. We take $\dim(F) = m+1$ and let $\mathbf{a} = (a_1,\ldots,a_{m+1})$ and $\mathbf{b} = (b_1,\ldots,b_{m+1})$. We need to show that

$$\nu_F\big((-\mathbf{a},\mathbf{b})^c\big) = \nu_G\Big(p_{GF}^{-1}\big((-\mathbf{a},\mathbf{b})^c\big)\Big)$$

for every $\mathbf{a},\mathbf{b} \in [\mathbf{0},\boldsymbol{\infty})^F \setminus \{\mathbf{0}_F\}$ with $(-\mathbf{a},\mathbf{b})^c$ and $p_{GF}^{-1}\big((-\mathbf{a},\mathbf{b})^c\big)$ both continuity sets. If $a_i = 0, b_i > 0$ (or $a_i > 0, b_i = 0$) for some $i \in \{1,\ldots,m+1\}$, then $\mathbf{0}_F \in (-\mathbf{a},\mathbf{b})^c$; therefore, one does not need to consider such cases. If $a_i = b_i = 0$ for some $i \in \{1,\ldots,m+1\}$, the statement is automatically true by induction hypothesis. Hence, it suffices to check the cases $a_i > 0, b_i > 0$ for all $i \in \{1,\ldots,m+1\}$. Then, the same argument as applied in the one-dimensional case finishes the proof. To complete (6.4) for any Borel set, consider $B \in \mathcal{B}(\mathbb{R}^F)$ with $\mathbf{0}_F \in B$. Since $\nu_F\big(\{\mathbf{0}_F\}\big) = 0$,

$$\nu_F(B) = \nu_F\big(B \setminus \{\mathbf{0}_F\}\big) = \nu_G\Big(p_{GF}^{-1}\big(B \setminus \{\mathbf{0}_F\}\big)\Big).$$

We have seen that a family of measures $(\nu_F,\ F \subseteq T, \text{finite})$, with each ν_F defined on $(\mathbb{R}^F, \mathcal{B}(\mathbb{R}^F))$, satisfies the same conditions as (6.2) (and hence (6.3)). Now, the Kolmogorov extension-like argument, which was essentially adopted in Proposition 1.1 of

Maruyama (1970), proves the existence of a cylindrical measure that fulfills (i) and (ii) in Theorem 6.1.1.

To prove uniqueness of a tail measure, we suppose that there exists another tail measure ρ on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^T)$, such that

$$\nu\Big(p_F^{-1}\big(B \setminus \{\mathbf{0}_F\}\big)\Big) = \rho\Big(p_F^{-1}\big(B \setminus \{\mathbf{0}_F\}\big)\Big), \;\; B \in \mathcal{B}(\mathbb{R}^F)\,, \tag{6.5}$$

for all finite sets $F \subseteq T$. In the sequel, we will prove that $\nu = \rho$. Since $\mathcal{B}(\mathbb{R})^T$ can be expressed as

$$\mathcal{B}(\mathbb{R})^T = \big\{p_S^{-1}(B) : B \in \mathcal{B}(\mathbb{R}^s),\, S \subseteq T \text{ is a countable set}\big\}\,, \tag{6.6}$$

it is enough to show that

$$\nu \circ p_S^{-1} = \rho \circ p_S^{-1} \quad \text{for any countable set } S \subseteq T\,. \tag{6.7}$$

By Monotone class theorem, it suffices to check (6.7) for all finite sets $F \subseteq T$. For every $B \in \mathcal{B}(\mathbb{R}^F)$,

$$\begin{aligned}\nu\big(p_F^{-1}(B)\big) &= \nu\Big(p_F^{-1}\big(B \setminus \{\mathbf{0}_F\}\big)\Big) + \nu\Big(p_F^{-1}\big(B \cap \{\mathbf{0}_F\}\big)\Big)\\ &= \rho\Big(p_F^{-1}\big(B \setminus \{\mathbf{0}_F\}\big)\Big) + \nu\Big(p_F^{-1}\big(\{\mathbf{0}_F\}\big)\Big)\mathbf{1}_{\{\mathbf{0}_F \in B\}}\,.\end{aligned}$$

Thus, (6.7) will be established if

$$\nu\Big(p_F^{-1}\big(\{\mathbf{0}_F\}\big)\Big) = \rho\Big(p_F^{-1}\big(\{\mathbf{0}_F\}\big)\Big)$$

for all finite $F \subseteq T$.

By condition (ii) in Theorem 6.1.1, there is a countable set $F \subseteq S \subseteq T$, such that

$$\begin{aligned}\nu\Big(p_F^{-1}\big(\{\mathbf{0}_F\}\big)\Big) &= \nu\Big(p_F^{-1}\big(\{\mathbf{0}_F\}\big) \setminus p_S^{-1}\big(\{\mathbf{0}_S\}\big)\Big)\,,\\ \rho\Big(p_F^{-1}\big(\{\mathbf{0}_F\}\big)\Big) &= \rho\Big(p_F^{-1}\big(\{\mathbf{0}_F\}\big) \setminus p_S^{-1}\big(\{\mathbf{0}_S\}\big)\Big)\,.\end{aligned}$$

Since S is countable, there exists a sequence of finite sets $F \subseteq G_n \uparrow S$ so that

$$\nu\Big(p_F^{-1}\big(\{\mathbf{0}_F\}\big)\Big) = \lim_{n\to\infty} \nu\Big(p_F^{-1}\big(\{\mathbf{0}_F\}\big) \setminus p_{G_n}^{-1}\big(\{\mathbf{0}_{G_n}\}\big)\Big) = \lim_{n\to\infty} \nu\Big(p_{G_n}^{-1}\big(p_{G_nF}^{-1}\big(\{\mathbf{0}_F\}\big) \setminus \{\mathbf{0}_{G_n}\}\big)\Big)\,.$$

Similarly, we get $\rho\Big(p_F^{-1}\big(\{\mathbf{0}_F\}\big)\Big) = \lim_{n\to\infty} \rho\Big(p_{G_n}^{-1}\big(p_{G_nF}^{-1}\big(\{\mathbf{0}_F\}\big) \setminus \{\mathbf{0}_{G_n}\}\big)\Big)$. Now, (6.5) finishes the proof. $\square$

Although a tail measure is not necessarily σ-finite, the next proposition provides a necessary and sufficient condition for it to be σ-finite.

Proposition 6.1.4. *Under the assumptions of Theorem 6.1.1, a tail measure ν is σ-finite if and only if there is a countable set $T_0 \subseteq T$, such that $\nu\Big(p_{T_0}^{-1}\big(\{\mathbf{0}_{T_0}\}\big)\Big) = 0$.*

Proof. Suppose, first, that ν is σ-finite. There is a sequence $(A_j) \subseteq \mathcal{B}(\mathbb{R})^T$ with $\mathbb{R}^T = \bigcup_{j=1}^{\infty} A_j$ and $\nu(A_j) < \infty$. In view of (6.6), each A_j can be written as $A_j = p_{S_j}^{-1}(B_j)$ for some countable $S_j \subseteq T$ and $B_j \in \mathcal{B}(\mathbb{R}^{S_j})$. Define a countable set $T_1 = \bigcup_{j=1}^{\infty} S_j$.

If $\mathbf{0}_{S_j} \in B_j$ for some $j \geq 1$, then $p_{T_1}^{-1}\big(\{\mathbf{0}_{T_1}\}\big) \subseteq p_{S_j}^{-1}(B_j)$ and, hence, $\nu\Big(p_{T_1}^{-1}\big(\mathbf{0}_{T_1}\big)\Big) \leq \nu(A_j) < \infty$. From condition (ii) in Theorem 6.1.1, it follows that

$$\nu\Big(p_{T_1}^{-1}\big(\{\mathbf{0}_{T_1}\}\big) \setminus p_{T_2}^{-1}\big(\{\mathbf{0}_{T_2}\}\big)\Big) = \nu\Big(p_{T_1}^{-1}\big(\{\mathbf{0}_{T_1}\}\big)\Big)$$

for some countable T_2 with $T_1 \subseteq T_2 \subseteq T$.

Now we get

$$\begin{aligned}\nu\Big(p_{T_2}^{-1}\big(\{\mathbf{0}_{T_2}\}\big)\Big) &= \nu\Big(p_{T_1}^{-1}\big(\{\mathbf{0}_{T_1}\}\big) \setminus \Big(p_{T_1}^{-1}\big(\{\mathbf{0}_{T_1}\}\big) \setminus p_{T_2}^{-1}\big(\{\mathbf{0}_{T_2}\}\big)\Big)\Big) \\ &= \nu\Big(p_{T_1}^{-1}\big(\{\mathbf{0}_{T_1}\}\big)\Big) - \nu\Big(p_{T_1}^{-1}\big(\{\mathbf{0}_{T_1}\}\big)\Big) = 0\,.\end{aligned}$$

On the contrary, if $\mathbf{0}_{S_j} \notin B_j$ for all $j \geq 1$,

$$\nu\Big(p_{T_1}^{-1}\big(\{\mathbf{0}_{T_1}\}\big)\Big) \leq \sum_{j=1}^{\infty} \nu\Big(p_{T_1}^{-1}\big(\{\mathbf{0}_{T_1}\}\big) \cap p_{S_j}^{-1}(B_j)\Big) = \sum_{j=1}^{\infty} \nu(\emptyset) = 0\,.$$

Conversely, assume that $\nu\Big(p_{T_0}^{-1}\big(\{\mathbf{0}_{T_0}\}\big)\Big) = 0$ for some countable $T_0 \subseteq T$. We can express $\mathbb{R}^T$ by

$$\mathbb{R}^T = \bigcup_{t \in T_0} \bigcup_{n=1}^{\infty} \Big(p_{T_0}^{-1}\big(\{\mathbf{0}_{T_0}\}\big) \cup \big\{x \in \mathbb{R}^T : |x_t| > n^{-1}\big\}\Big)\,.$$

Since

$$\begin{aligned}&\nu\Big(p_{T_0}^{-1}\big(\{\mathbf{0}_{T_0}\}\big) \cup \big\{x \in \mathbb{R}^T : |x_t| > n^{-1}\big\}\Big) \\ &= \nu\big(x \in \mathbb{R}^T : |x_t| > n^{-1}\big) = \mu_t\big(y : |y| > n^{-1}\big) < \infty\,,\end{aligned}$$

ν is σ-finite. $\square$

The next proposition describes the homogeneity property of a tail measure. It can be proved directly by the homogeneity of finite-dimensional Radon measures in (6.1). So, we only presents the result.

Proposition 6.1.5. *Under the assumptions of Theorem 6.1.1, a tail measure ν has the homogeneity property. That is, there exists an $\alpha > 0$ such that*

$$\nu(cA) = c^{-\alpha}\nu(A) \quad \textit{for all } A \in \mathcal{B}(\mathbb{R})^T,\ c > 0\,.$$

6.2 Examples

This section presents the tail measures of several stochastic processes; moving averages, independent processes, stochastic volatility processes, and GARCH processes.

Example 6.2.1. Let $T = \mathbb{Z}$ or $\mathbb{R}$, and $(X_t,\, t \in T)$ be a stochastic process with integral representation

$$X_t = \int_E f_t(x)dM(x), \quad t \in T\,, \tag{6.8}$$

where M is an independently scattered infinitely divisible random measure on a measurable space $(E, \mathcal{E})$ with σ-finite control measure m and a local Lévy measure $\rho(s, \cdot)$, $s \in E$. The functions f_t are deterministic functions of the form $f_t(x) = f \circ \psi_t(x)$, $x \in E$, $t \in T$, where $f : E \to \mathbb{R}$ is a measurable function, and $\psi_t : E \to E$, $t \in T$ is a family of measurable maps. Rajput and Rosiński (1989) describes a condition under which X_t is well-defined, which will be assumed throughout this example. Then $(X_t,\, t \in T)$ is, automatically, an well-defined infinitely divisible process. The function level Lévy measure of $(X_t,\, t \in T)$ is given by $(\rho \times m) \circ h^{-1}$, where $h(x, s) = xf_\cdot(s)$, $x \in \mathbb{R}$, $s \in E$. Furthermore, we suppose the following conditions:

- There exists a measurable and regularly varying function $H : (0, \infty) \to (0, \infty)$ of index $-\alpha$ for some $\alpha > 0$. There also exist measurable functions $w_\pm : E \to [0, \infty)$

such that for every $s \in E$,

$$\lim_{u\to\infty} \frac{\rho\big(s,(u,\infty)\big)}{H(u)} = w_+(s) \quad \text{and} \quad \lim_{u\to\infty} \frac{\rho\big(s,(-\infty,-u)\big)}{H(u)} = w_-(s)\,. \tag{6.9}$$

- The convergence above is uniform: there exists $u_0 > 0$ with

$$\sup_{u\ge u_0} \frac{\rho\big(s,(u,\infty)\big)}{H(u)} \le 2w_+(s) \quad \text{and} \quad \sup_{u\ge u_0} \frac{\rho\big(s,(-\infty,-u)\big)}{H(u)} \le 2w_-(s) \tag{6.10}$$

  for all $s \in E$.

- $f : E \to \mathbb{R}$ is bounded on E and, for some $\xi \in (0,\alpha)$,

$$\int_E w_\pm(s)|f_t(s)|^{\alpha-\xi} m(ds) < \infty \tag{6.11}$$

  for all $t \in T$.

Then, one can show that $(X_t,\, t \in T)$ has regularly varying tails with index α and the tail measure of $(X_t,\, t \in T)$ is given by

$$\nu = (\rho_* \times m) \circ h^{-1}\,,$$

where

$$\rho_*(s,dx) = w_+(s)\frac{\alpha}{x^{1+\alpha}}\mathbf{1}_{\{x>0\}}dx + w_-(s)\frac{\alpha}{|x|^{1+\alpha}}\mathbf{1}_{\{x<0\}}dx\,. \tag{6.12}$$

To show this, we only have to prove that as $u \to \infty$,

$$H(u)^{-1}P\big((X_{t_1},\dots,X_{t_k}) \in u\,\cdot\big) \xrightarrow{v} (\rho_* \times m)\Big\{(x,s) : \big(xf_{t_1}(s),\dots,xf_{t_k}(s)\big) \in \cdot\Big\}$$

vaguely in $\overline{\mathbb{R}}^k \setminus \{\mathbf{0}\}$ for all $t_1,\dots,t_k \in T$, $k \ge 1$.

Equivalently, we need prove that for all $a_i > 0$ and $e_i \in \{-1,1\}$, $i = 1,\dots,k$,

$$H(u)^{-1}P\big(e_iX_{t_i} > a_iu,\ \ i=1,\dots,k\big) \to (\rho_* \times m)\big\{(x,s) : xe_if_{t_i}(s) > a_i,\ \ i=1,\dots,k\,\big\}\,. \tag{6.13}$$

The tail behavior of the probability law of $(X_t,\, t \in T)$ is known to coincide with that of the function level Lévy measure of $(X_t,\, t \in T)$. See Theorem 2.1 in Rosiński and Samorodnitsky (1993). Thus, (6.13) is equivalent to

$$H(u)^{-1}(\rho \times m)\big\{(x,s) : xe_if_{t_i}(s) > a_iu,\ \ i=1,\dots,k\,\big\} \tag{6.14}$$

$$\to (\rho_* \times m)\big\{(x,s) : x e_i f_{t_i}(s) > a_i, \ \ i = 1, \ldots, k\,\big\}\,.$$

The left hand side of (6.14) is equal to

$$\int_{\{e_i f_{t_i}(s)>0,\ i=1,\ldots,k\}} H(u)^{-1}\rho\Big(s, \big(u \max_{1\le i\le k} a_i |f_{t_i}(s)|^{-1}, \infty\big)\Big) m(ds)$$

$$+ \int_{\{e_i f_{t_i}(s)<0,\ i=1,\ldots,k\}} H(u)^{-1}\rho\Big(s, \big(-\infty, -u \max_{1\le i\le k} a_i |f_{t_i}(s)|^{-1}\big)\Big) m(ds)\,.$$

On the other hand, the right hand side of (6.14) is equal to

$$\int_{\{e_i f_{t_i}(s)>0,\ i=1,\ldots,k\}} w_+(s) \min_{1\le i\le k} \left(\frac{|f_{t_i}(s)|}{a_i}\right)^{\alpha} m(ds)$$

$$+ \int_{\{e_i f_{t_i}(s)<0,\ i=1,\ldots,k\}} w_-(s) \min_{1\le i\le k} \left(\frac{|f_{t_i}(s)|}{a_i}\right)^{\alpha} m(ds)\,.$$

Due to their symmetric structure, it suffices to check the convergence of the integral defined on $\{e_i f_{t_i}(s) > 0,\ i = 1, \ldots, k\}$. By condition (6.9),

$$H(u)^{-1}\rho\Big(s, \big(u \max_{1\le i\le k} a_i |f_{t_i}(s)|^{-1}, \infty\big)\Big) \to w_+(s) \min_{1\le i\le k} \left(\frac{|f_{t_i}(s)|}{a_i}\right)^{\alpha}$$

for every $s \in E$. Therefore, we only need to justify taking the limit inside. In order to apply the dominated convergence theorem, we must find a nonnegative function $K \in L^1(E, m)$ such that

$$H(u)^{-1}\rho\Big(s, \big(u \max_{1\le i\le k} a_i |f_{t_i}(s)|^{-1}, \infty\big)\Big) \le K(s)$$

for every $s \in E$ and sufficiently large $u > 0$.

In view of uniformity condition (6.10), for all $u \ge u_0 \sup_{s\in E} |f(s)| / \max_{1\le i\le k} a_i$ (the right hand side is finite, since f is bounded),

$$H(u)^{-1}\rho\Big(s, \big(u \max_{1\le i\le k} a_i |f_{t_i}(s)|^{-1}, \infty\big)\Big) \le 2w_+(s) \frac{H\Big(u \max_{1\le i\le k} a_i |f_{t_i}(s)|^{-1}\Big)}{H(u)}\,.$$

The Potter bounds (see e.g. Proposition 0.8 in Resnick (1987)) provide, for some $C_i > 0$, $i = 1, 2$,

$$\frac{H\Big(u \max_{1\le i\le k} a_i |f_{t_i}(s)|^{-1}\Big)}{H(u)} \le C_1 \left(\max_{1\le i\le k} \frac{a_i}{|f_{t_i}(s)|}\right)^{-\alpha+\xi} \le C_2 \sum_{i=1}^{k} |f_{t_i}(s)|^{\alpha-\xi}$$

for all u large enough. Now, because of (6.11), an appropriate $L^1(E, m)$-upper bound $K(\cdot)$ is easy to take and the proof is complete.

Example 6.2.2. We will consider, once again, the process (6.8), but here, a local Lévy measure ρ is independent of $s \in E$. We assume that ρ has a balanced regularly varying tail: for some $\alpha > 0$, $\rho\big(x : |x| > \cdot\big)$ is regularly varying with index $-\alpha$, and

$$\frac{\rho(y,\infty)}{\rho\big(x : |x| > y\big)} \to p, \quad \frac{\rho(-\infty,-y)}{\rho\big(x : |x| > y\big)} \to q \tag{6.15}$$

as $y \to \infty$, where $0 \le p,\, q \le 1$ with $p + q = 1$.

In this example, we remove boundedness assumption of f, and instead, the following integrability condition is assumed: there exists $0 < \beta \le 2$ such that

$$\begin{cases} \int_E |f_t(s)|^{\alpha-\xi} \vee |f_t(s)|^{\beta} m(ds) < \infty & \text{for some } 0 < \xi < \beta - \alpha \ \text{ if } 0 < \alpha < \beta\,, \\ \int_E |f_t(s)|^{\alpha-\xi} \vee |f_t(s)|^{\alpha+\xi} m(ds) < \infty & \text{for some } 0 < \xi < \alpha \ \text{ if } \alpha \ge \beta\,, \end{cases}$$

for all $t \in T$. If $\beta \neq 2$, the lower tail behavior of ρ has to be specified explicitly, that is,

$$y^\beta \rho\big(x : |x| > y\big) \to 0 \quad \text{as } y \downarrow 0\,.$$

Under these assumptions, $(X_t,\, t \in T)$ has, once again, regularly varying tails with index α and its tail measure is given by

$$\nu = (\rho_* \times m) \circ h^{-1},$$

where h is the same function as before, and

$$\rho_*(dx) = p\frac{\alpha}{x^{1+\alpha}} \mathbf{1}_{\{x>0\}} dx + q\frac{\alpha}{|x|^{1+\alpha}} \mathbf{1}_{\{x<0\}} dx\,. \tag{6.16}$$

For the proof, let $H(u) = \rho\big(x : |x| > u\big)$. By the same argument as Example 6.2.1, it suffices to verify that, for all $t_1, \dots, t_k \in T$, $k \ge 1$, $a_i > 0$ and $e_i \in \{-1, 1\}$, $i = 1, \dots, k$,

$$\int_{\{e_i f_{t_i}(s) > 0,\ i=1,\dots,k\}} H(u)^{-1} \rho\big(u \max_{1 \le i \le k} a_i |f_{t_i}(s)|^{-1}, \infty\big) m(ds)$$
$$\to \int_{\{e_i f_{t_i}(s) > 0,\ i=1,\dots,k\}} p \min_{1 \le i \le k} \left(\frac{|f_{t_i}(s)|}{a_i}\right)^\alpha m(ds)\,,$$
$$\int_{\{e_i f_{t_i}(s) < 0,\ i=1,\dots,k\}} H(u)^{-1} \rho\big(-\infty, -u \max_{1 \le i \le k} a_i |f_{t_i}(s)|^{-1}\big) m(ds)$$

$$\to \int_{\{e_i f_{t_i}(s)<0,\ i=1,\dots,k\}} q \min_{1\le i\le k} \left(\frac{|f_{t_i}(s)|}{a_i}\right)^\alpha m(ds)\,.$$

By regular variation of $H(u)$, we have, as $u\to\infty$,

$$H(u)^{-1}\rho\big(u \max_{1\le i\le k} a_i|f_{t_i}(s)|^{-1},\infty\big) \to p \min_{1\le i\le k}\left(\frac{|f_{t_i}(s)|}{a_i}\right)^\alpha,$$

$$H(u)^{-1}\rho\big(-\infty,-u \max_{1\le i\le k} a_i|f_{t_i}(s)|^{-1}\big) \to q \min_{1\le i\le k}\left(\frac{|f_{t_i}(s)|}{a_i}\right)^\alpha$$

for any $s\in E$. It remains to find a measurable function $K\in L^1(E,m)$, such that

$$H(u)^{-1}\rho\big(x:|x|>u\max_{1\le i\le k} a_i|f_{t_i}(s)|^{-1}\big)\le K(s)$$

for any $s\in E$ and sufficiently large $u>0$.

We see from the Potter bounds that, for some $C_i>0$, $i=1,2$,

$$\frac{\rho\big(x:|x|>u\max_{1\le i\le k} a_i|f_{t_i}(s)|^{-1}\big)}{\rho\big(x:|x|>u\big)}\mathbf{1}\Big(u\max_{1\le i\le k} a_i|f_{t_i}(s)|^{-1}>1\Big)$$

$$\le C_1\left\{\left(\max_{1\le i\le k}\frac{a_i}{|f_{t_i}(s)|}\right)^{-\alpha+\xi}+\left(\max_{1\le i\le k}\frac{a_i}{|f_{t_i}(s)|}\right)^{-\alpha-\xi}\right\}$$

$$\le C_2\sum_{i=1}^k\big\{|f_{t_i}(s)|^{\alpha-\xi}+|f_{t_i}(s)|^{\alpha+\xi}\big\}$$

for all u large enough.

Since $y^\beta\rho\big(x:|x|>y\big)\to 0$ as $y\downarrow 0$, there exists $C_3>0$ with $\rho\big(x:|x|>y\big)<C_3y^{-\beta}$ for all $0<y\le 1$. Thus, we see that

$$\frac{\rho\big(x:|x|>u\max_{1\le i\le k} a_i|f_{t_i}(s)|^{-1}\big)}{\rho\big(x:|x|>u\big)}\mathbf{1}\Big(u\max_{1\le i\le k} a_i|f_{t_i}(s)|^{-1}\le 1\Big)$$

$$\le\frac{C_4}{u^\beta\rho\big(x:|x|>u\big)}\sum_{i=1}^k|f_{t_i}(s)|^\beta\,\mathbf{1}\Big(u\max_{1\le i\le k} a_i|f_{t_i}(s)|^{-1}\le 1\Big)$$

for some $C_4>0$. If $0<\alpha<\beta$, then $u^\beta\rho\big(x:|x|>u\big)\to\infty$ as $u\to\infty$ and, hence,

$$\frac{\rho\big(x:|x|>u\max_{1\le i\le k} a_i|f_{t_i}(s)|^{-1}\big)}{\rho\big(x:|x|>u\big)}\mathbf{1}\Big(u\max_{1\le i\le k} a_i|f_{t_i}(s)|^{-1}\le 1\Big)\le C_4\sum_{i=1}^k|f_{t_i}(s)|^\beta$$

for all u large enough.

On the contrary, if $\alpha\ge\beta$, there is $C_5>0$ such that

$$\frac{1}{u^\beta\rho\big(x:|x|>u\big)}\le C_5u^{\alpha-\beta+\xi}$$

for all u large enough. Therefore, for some $C_6 > 0$,

$$\frac{\rho\big(x : |x| > u \max_{1\le i\le k} a_i|f_{t_i}(s)|^{-1}\big)}{\rho\big(x : |x| > u\big)}\,\mathbf{1}\Big(u \max_{1\le i\le k} a_i|f_{t_i}(s)|^{-1} \le 1\Big)$$

$$\le C_4C_5u^{\alpha-\beta+\xi}\sum_{i=1}^{k}|f_{t_i}(s)|^{\beta}\,\mathbf{1}\Big(u \max_{1\le i\le k} a_i|f_{t_i}(s)|^{-1} \le 1\Big) \le C_6\sum_{i=1}^{k}|f_{t_i}(s)|^{\alpha+\xi}.$$

In either case, we have found an $L^1(E,m)$-function $K(\cdot)$ as desired.

Example 6.2.3. Let T be an arbitrary set and $\mathbf{X} = (X_t,\, t \in T)$ be an independent process (i.e., for all $t_1,\dots,t_k \in T$, $X_{t_1},\dots,X_{t_k}$ are independent). Suppose that there is a regularly varying function $H : (0,\infty) \to (0,\infty)$ with index $-\alpha$ for some $\alpha > 0$, such that for every $t \in T$, there is a Radon measure μ_t on $\overline{\mathbb{R}} \setminus \{\mathbf{0}\}$ with $\mu(\overline{\mathbb{R}} \setminus \mathbb{R}) = 0$, such that, as $u \to \infty$,

$$H(u)^{-1}P\big(X_t \in u\,\cdot\big) \xrightarrow{v} \mu_t(\cdot) \tag{6.17}$$

vaguely in $\overline{\mathbb{R}} \setminus \{\mathbf{0}\}$. Assume that at least one Radon measure μ_{t_0}, $t_0 \in T$, is a nonzero measure. Then the tail measure of $\mathbf{X}$ is given by

$$\nu(A) = \sum_{t\in T}\nu_t(A),\quad A \in \mathcal{B}(\mathbb{R})^T,$$

where

$$\nu_t(A) = \mu_t\big((1,\infty)\big)\int_0^\infty \alpha y^{-(\alpha+1)}\mathbf{1}_{\{ye(t)\in A\}}dy + \mu_t\big((-\infty,-1)\big)\int_{-\infty}^0 \alpha|y|^{-(\alpha+1)}\mathbf{1}_{\{ye(t)\in A\}}dy\,,$$

$$e(t) \in \mathbb{R}^T \text{ with } e(t)|_s = \begin{cases} 1 & \text{if } s = t\,, \\ 0 & \text{otherwise.}\end{cases}$$

To see this, the multivariate regular variation of $\big(X_{t_1},\dots,X_{t_k}\big)$ is derived from (6.17) and independence of $\big(X_{t_1},\dots,X_{t_k}\big)$ (see e.g. Lemma 7.2 in Resnick (2007)):

$$H(u)^{-1}P\big((X_{t_1},\dots,X_{t_k}) \in u\,\cdot\big) \xrightarrow{v} \sum_{j=1}^{k}\big(\epsilon_0 \times \cdots \times \mu_{t_j} \times \cdots \times \epsilon_0\big)$$

vaguely in $\overline{\mathbb{R}}^k \setminus \{\mathbf{0}\}$. Here

$$\epsilon_0(A) = \begin{cases} 1 & \text{if } 0 \in A\,, \\ 0 & \text{otherwise.}\end{cases}$$

It is easy to verify that, with $F = \{t_1, \ldots, t_k\}$,

$$\begin{cases} \nu_{t_j} \circ p_F^{-1} = \epsilon_0 \times \cdots \times \mu_{t_j} \times \cdots \times \epsilon_0\,, \quad j = 1, \ldots, k\,, \\ \nu_t \circ p_F^{-1} = 0 \quad \text{if } \ t \notin F\,. \end{cases}$$

Therefore,

$$\sum_{t \in T} \nu_t \circ p_F^{-1} = \sum_{j=1}^{k} \bigl(\epsilon_0 \times \cdots \times \mu_{t_j} \times \cdots \times \epsilon_0\bigr)\,.$$

Since the choice of a finite index set F is arbitrary, the tail measure of $\mathbf{X}$ turns out to be $\nu = \sum_{t \in T} \nu_t$.

Example 6.2.4. Let $(X_n,\, n = 1, 2, \ldots)$ be a simple stochastic volatility process of the form

$$X_n = \sigma_n Z_n, \quad n = 1, 2, \ldots,$$

where (σ_n) is a nonnegative stationary sequence, representing volatility. (Z_n) is a sequence of i.i.d. random variables and is independent of (σ_n), and (Z_n) has a regularly varying tail with index α. Let μ denote the limiting Radon measure of the regular variation for (Z_n). Assume that volatility sequence (σ_n) has a significantly lighter tail than that of (Z_n); that is, (σ_n) has finite $(\alpha+\epsilon)$th moment for some $\epsilon > 0$. Then, their product $\mathbf{X} = (X_n,\, n = 1, 2, \ldots)$ becomes a stationary sequence with regularly varying tail of the same index α, and the tail measure of $\mathbf{X}$ is given by

$$\nu(A) = E\bigl(\sigma^\alpha\bigr) \sum_{j=1}^{\infty} \nu_j(A)$$

for a Borel set A, where

$$\nu_j(A) = \mu\bigl((1, \infty)\bigr) \int_0^\infty \alpha y^{-(\alpha+1)} \mathbf{1}_{\{y e(j) \in A\}} dy + \mu\bigl((-\infty, -1)\bigr) \int_{-\infty}^0 \alpha |y|^{-(\alpha+1)} \mathbf{1}_{\{y e(j) \in A\}} dy\,.$$

To see this, since $E\sigma^{\alpha+\epsilon} < \infty$, the multivariate Breiman's theorem (see e.g. Basrak et al. (2002b)) yields

$$\begin{aligned} &H(u)^{-1} P\bigl((\sigma_1 Z_1, \ldots, \sigma_k Z_k) \in u\,\cdot\bigr) \\ &\xrightarrow{v} \sum_{j=1}^{k} E\left[\bigl(\epsilon_0 \times \cdots \times \mu \times \cdots \times \epsilon_0\bigr)\Bigl\{\mathbf{x} : \bigl(\sigma_1 x_1, \ldots, \sigma_k x_k\bigr) \in \cdot\Bigr\}\right] \end{aligned}$$

vaguely in $\overline{\mathbb{R}}^k \setminus \{\mathbf{0}\}$ for every $k \geq 1$. Because of the stationarity of (σ_n) and the homogeneity property of μ,

$$\sum_{j=1}^{k} E\left[\big(\epsilon_0 \times \cdots \times \mu \times \cdots \times \epsilon_0\big)\Big\{\mathbf{x} : \big(\sigma_1 x_1, \dots, \sigma_k x_k\big) \in \cdot\Big\}\right]$$

$$= E\big(\sigma^\alpha\big) \sum_{j=1}^{k} \big(\epsilon_0 \times \cdots \times \mu \times \cdots \times \epsilon_0\big)\,.$$

The same argument as in Example 6.2.3 establishes

$$\sum_{j=1}^{k} \big(\epsilon_0 \times \cdots \times \mu \times \cdots \times \epsilon_0\big) = \sum_{j=1}^{\infty} \nu_j \circ p_{\{1\dots k\}}^{-1}\,,$$

which finishes the proof.

If $Z_n = Z_1$ for all $n = 1, 2, \dots$, then Z_1 works as a common heavy-tailed component for (X_n), which means that (X_n) is expected to exhibit a longer memory than the previous i.i.d. setup. In this case, the tail measure of $\mathbf{X}$ can be simply written as

$$E\mu\big\{x : x(\sigma_1, \sigma_2, \dots)' \in \cdot\big\}\,.$$

Example 6.2.5. This example considers the tail measure of GARCH processes. Regular variation of GARCH processes was rigorously discussed by Basrak et al. (2002b) from the viewpoint of stochastic recurrence equations. A nice review on heavy-tailed GARCH processes, including continuous-time models, is provided by Fasen (2010). In order to calculate the tail measure explicitly, we will concentrate on GARCH$(1,1)$ processes and the argument is, to some extent, parallel to that of Davis and Mikosch (2009). Specifically, we will consider the following GARCH$(1,1)$ process:

$$X_n = \sigma_n Z_n\,, \quad n \in \mathbb{N}\,, \tag{6.18}$$

$$\sigma_n^2 = \alpha_0 + \alpha_1 X_{n-1}^2 + \beta_1 \sigma_{n-1}^2\,, \tag{6.19}$$

where α_0, α_1 and β_1 are positive constants, σ_0 is a nonnegative random variable, and (Z_n) are i.i.d. symmetric random variables with unit variance. Let $A_n = \alpha_1 Z_{n-1}^2 + \beta_1$, and suppose that $E \log A = E \log\big(\alpha_1 Z^2 + \beta_1\big) < 0$. Under such a circumstance, there exists a stationary solution (X_n, σ_n) to stochastic equations (6.18) and (6.19); see Babillot et al. (1997) for more details. Assume, additionally, that the law of $\log A$ is nonarithmetic, $P(A > 1) > 0$ and there exists $1 < h_0 \leq \infty$, such that

$$EA^h < \infty \text{ for all } h < h_0 \text{ and } EA^{h_0} = \infty \,.$$

Then, some constant $\alpha > 0$ satisfies $EA^{\alpha/2} = 1$ and, further, (σ_n) is regularly varying with index α (see Mikosch and Stărică (2000) for a detailed proof). We denote by μ the limiting Radon measure for the regular variation of σ_n; namely, there is a function $H : (0, \infty) \to (0, \infty)$ such that

$$H(u)^{-1} P\big(\sigma_n \in u \,\cdot\big) \xrightarrow{v} \mu(\cdot)$$

vaguely in $\overline{\mathbb{R}} \setminus \{0\}$. Then, a stationary sequence (X_n) can be proved to have regularly varying tails with the same index α, and one can also see that the tail measure of (X_n) is given by

$$E\mu\Big\{x : x\big(Z_0, Z_1\sqrt{A_1}, Z_2\sqrt{A_1 A_2}, \dots\big) \in \cdot\Big\} \,.$$

For the proof, fix $(a_0, \dots, a_k) \in [\mathbf{0}, \boldsymbol{\infty})^{k+1} \setminus \{\mathbf{0}\}$ and $e_i \in \{-1, 1\}$, $i = 0, \dots, k$. Lemma 2.1 in Davis and Mikosch (2009) gives a useful approximation of $(X_0, \dots, X_k)$:

$$(X_0, X_1, \dots, X_k) = \sigma_0\big(Z_0, Z_1\sqrt{A_1}, \dots, Z_k\sqrt{A_1 A_2 \dots A_k}\big) + \mathbf{R} \,,$$

where $H(u)^{-1} P\big(\|\mathbf{R}\| > u\epsilon\big) \to 0$, as $u \to \infty$, for every $\epsilon > 0$.

Thus, as $u \to \infty$, we have

$$H(u)^{-1} P\big(e_i X_i > u a_i, \;\; i = 0, \dots, k\big)$$

$$\sim H(u)^{-1} P\big(\sigma_0 e_0 Z_0 > u a_0, \; \sigma_0 e_1 Z_1\sqrt{A_1} > u a_1, \dots, \sigma_0 e_k Z_k\sqrt{A_1 \dots A_k} > u a_k\big) \,.$$

Since $E\big(Z_i\sqrt{A_1 \dots A_i}\big)^{\alpha+\epsilon} < \infty$ for ϵ small enough, an application of the multivariate Breiman's theorem yields

$$H(u)^{-1} P\big(\sigma_0 e_0 Z_0 > u a_0, \; \sigma_0 e_1 Z_1\sqrt{A_1} > u a_1, \dots, \sigma_0 e_k Z_k\sqrt{A_1 \dots A_k} > u a_k\big)$$

$$\to E\mu\big\{x : x e_0 Z_0 > a_0, \; x e_1 Z_1\sqrt{A_1} > a_1, \dots, x e_k Z_k\sqrt{A_1 \dots A_k} > a_k\big\}$$

as required.

6.3 Connection between Tail Measures and Other Related Notions

This section investigates the relation between tail measures and their related notions. The tail measure turns out be a more comprehensive notion than those alternatives.

First of all, for a stationary sequence $(X_n,\, n \geq 0)$, the relation between the tail measure ν and upper tail dependence coefficient (1.6) is clearly described by

$$\lambda(n) = \frac{\nu\big\{\mathbf{x} \in \mathbb{R}^{\mathbb{N}} : \min(x_0, x_n) > 1\big\}}{\nu\big\{\mathbf{x} \in \mathbb{R}^{\mathbb{N}} : x_0 > 1\big\}}\,.$$

Let $(X_t,\, t \in T)$, $T = \mathbb{R}$ or $\mathbb{Z}$ be a stationary process with tail measure ν. Fasen (2010) studied extreme dependence measure defined by

$$\underline{\chi}_{(t_1 \dots t_d)}(y_1, \dots, y_d) = \lim_{u\to\infty} H(u)^{-1} P\big(X_{t_1} > uy_1, \dots, X_{t_d} > uy_d\big)\,,$$

$$\overline{\chi}_{(t_1 \dots t_d)}(y_1, \dots, y_d) = \lim_{u\to\infty} H(u)^{-1} P\big(X_{t_1} > uy_1 \text{ or} \dots \text{ or } X_{t_d} > uy_d\big)$$

for some regularly varying function $H : (0,\infty) \to (0,\infty)$. One can obtain an obvious relation

$$\underline{\chi}_{(t_1 \dots t_d)}(y_1, \dots, y_d) = \nu\big\{\mathbf{x} \in \mathbb{R}^T : x_{t_j} > y_j,\;\; j = 1, \dots, d\,\big\}\,.$$

The connection between ν amd $\overline{\chi}$ can also be formulated easily by the inclusion-exclusion property. Let $(\mathbf{X}_n,\, n \geq 0)$ be a stationary process in $\mathbb{R}^d$. The tail measure ν then relates to extremogram (1.7) and (1.8) in such a way that

$$\gamma_{AB}(n) = \nu\big\{\mathbf{x} \in (\mathbb{R}^d)^{\mathbb{N}} : \mathbf{x}_0 \in A,\, \mathbf{x}_n \in B\big\}\,,$$

$$\rho_{AB}(n) = \frac{\nu\big\{\mathbf{x} \in (\mathbb{R}^d)^{\mathbb{N}} : \mathbf{x}_0 \in A,\, \mathbf{x}_n \in B\big\}}{\nu\big\{\mathbf{x} \in (\mathbb{R}^d)^{\mathbb{N}} : \mathbf{x}_0 \in A\big\}}\,,$$

where $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,d})$, $i \geq 0$ and both A and $A \times B$ are Borel sets bounded away from zero.

In relation to the examples in the preceding section, Fasen (2010) calculated the upper tail dependence coefficient and the extreme dependence measure of the process in Example 6.2.5. The extremograms for the processes in Examples 6.2.4 and 6.2.5 are provided by Davis and Mikosch (2009). Moreover, it is not difficult to calculate these quantities for the infinitely divisible processes in Examples 6.2.1 and 6.2.2.

We will consider a multivariate stationary time-series $\mathbf{X} = (\mathbf{X}_n,\, n \in \mathbb{Z})$ in $\mathbb{R}^d$ with regularly varying tails of index $\alpha > 0$. Basrak and Segers (2009) defined a limiting process $\mathbf{Y} = (\mathbf{Y}_n,\, n \in \mathbb{Z})$ in $\mathbb{R}^d$, called tail process, by

$$P\Big(\big(\mathbf{X}_m, \ldots, \mathbf{X}_n\big) \in u\,\cdot \;\big|\; \|\mathbf{X}_0\| > u\Big) \to P\Big(\big(\mathbf{Y}_m, \ldots, \mathbf{Y}_n\big) \in \,\cdot\,\Big)$$

weakly in $\mathbb{R}^{d(n-m+1)}$ for all $m, n \in \mathbb{Z}$ with $m \leq n$. Here $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^d$. On the other hand, the tail measure ν of $\mathbf{X}$ satisfies

$$P\Big(\big(\mathbf{X}_m, \ldots, \mathbf{X}_n\big) \in u\,\cdot \;\big|\; \|\mathbf{X}_0\| > u\Big) \to \frac{\nu\Big\{\mathbf{x} \in (\mathbb{R}^d)^{\mathbb{Z}} : \big(\mathbf{x}_m, \ldots, \mathbf{x}_n\big) \in \,\cdot\,,\ \|\mathbf{x}_0\| > 1\Big\}}{\nu\big\{\mathbf{x} \in (\mathbb{R}^d)^{\mathbb{Z}} :\ \|\mathbf{x}_0\| > 1\big\}}\,.$$

Therefore, we conclude

$$P(\mathbf{Y} \in \cdot\,) = \frac{\nu\big\{\mathbf{x} \in (\mathbb{R}^d)^{\mathbb{Z}} :\ \mathbf{x} \in \,\cdot\,,\ \|\mathbf{x}_0\| > 1\big\}}{\nu\big\{\mathbf{x} \in (\mathbb{R}^d)^{\mathbb{Z}} :\ \|\mathbf{x}_0\| > 1\big\}}\,.$$

Basrak and Segers (2009) also defined the spectral process of $\mathbf{X}$ by $\mathbf{\Theta}_n = \mathbf{Y}_n/\|\mathbf{Y}_0\|$, $n \in \mathbb{Z}$. Because of their Corollary 3.2, we find that $\mathbf{\Theta} = \big(\mathbf{\Theta}_n,\, n \in \mathbb{Z}\big)$ fulfills

$$P(\mathbf{\Theta} \in \cdot\,) = \frac{\nu\big\{\mathbf{x} \in (\mathbb{R}^d)^{\mathbb{Z}} :\ \mathbf{x}/\|\mathbf{x}_0\| \in \,\cdot\,,\ \|\mathbf{x}_0\| > 1\big\}}{\nu\big\{\mathbf{x} \in (\mathbb{R}^d)^{\mathbb{Z}} :\ \|\mathbf{x}_0\| > 1\big\}}\,.$$

The two most important properties of the tail process and the spectral process of $\mathbf{X}$ are given in Theorem 3.1 in Basrak and Segers (2009). For instance, statement (iii) of that theorem says that for all $i, m, n \in \mathbb{Z}$ with $m \leq 0 \leq n$ and for all bounded and continuous $f : (\mathbb{R}^d)^{n-m+1} \to \mathbb{R}$,

$$E\Big[f\big(\mathbf{\Theta}_{m-i}, \ldots, \mathbf{\Theta}_{n-i}\big)\Big] = E\left[f\left(\frac{\mathbf{\Theta}_m}{\|\mathbf{\Theta}_i\|}, \ldots, \frac{\mathbf{\Theta}_n}{\|\mathbf{\Theta}_i\|}\right)\|\mathbf{\Theta}_i\|^{\alpha}\right]. \tag{6.20}$$

Exploiting some *nice properties* of tail measures, we can provide a more natural alternative proof of (6.20). First, recall that the tail measure ν possesses homogeneity property as mentioned in Proposition 6.1.5. The second nice property is that due to the stationarity of $\mathbf{X}$, ν is shift invariant, that is,

$$\nu \circ \phi_n^{-1} = \nu \quad \text{for every } n \in \mathbb{Z}\,,$$

where $\phi_n : (\mathbb{R}^d)^{\mathbb{Z}} \to (\mathbb{R}^d)^{\mathbb{Z}}$ is the shift operator defined by

$$\phi_n(\ldots \mathbf{x}_{-1}, \mathbf{x}_0, \mathbf{x}_1 \ldots) = (\ldots \mathbf{x}_{n-1}, \mathbf{x}_n, \mathbf{x}_{n+1} \ldots)\,.$$

For the proof of (6.20), suppose for notational ease that $\nu\big\{\mathbf{x} \in (\mathbb{R}^d)^{\mathbb{Z}} : \ \|\mathbf{x}_0\| \ > 1\big\} = 1$. Using the identity

$$\frac{1}{\|\mathbf{x}_{-i}\|^{\alpha}} \int_0^{\|\mathbf{x}_{-i}\|} \alpha u^{\alpha-1} du = 1\,,$$

we write

$$\begin{aligned} &E\Big[f\big(\boldsymbol{\Theta}_{m-i}, \ldots, \boldsymbol{\Theta}_{n-i}\big)\Big] \\ &= \int_0^{\infty} \int_{(\mathbb{R}^d)^{\mathbb{Z}}} f\left(\frac{\mathbf{x}_{m-i}}{\|\mathbf{x}_0\|}, \ldots, \frac{\mathbf{x}_{n-i}}{\|\mathbf{x}_0\|}\right) \frac{\alpha u^{\alpha-1}}{\|\mathbf{x}_{-i}\|^{\alpha}} \mathbf{1}_{\{\|\mathbf{x}_{-i}\|>u,\, \|\mathbf{x}_0\|>1\}} \nu(\mathbf{dx}) du\,. \end{aligned}$$

By virtue of the shift invariance and the homogeneity property of ν,

$$\begin{aligned} &\int_0^{\infty} \int_{(\mathbb{R}^d)^{\mathbb{Z}}} f\left(\frac{\mathbf{x}_{m-i}}{\|\mathbf{x}_0\|}, \ldots, \frac{\mathbf{x}_{n-i}}{\|\mathbf{x}_0\|}\right) \frac{\alpha u^{\alpha-1}}{\|\mathbf{x}_{-i}\|^{\alpha}} \mathbf{1}_{\{\|\mathbf{x}_{-i}\|>u,\, \|\mathbf{x}_0\|>1\}} \nu(\mathbf{dx}) du \\ &= \int_0^{\infty} \int_{(\mathbb{R}^d)^{\mathbb{Z}}} f\left(\frac{\mathbf{x}_{m-i}}{\|\mathbf{x}_0\|}, \ldots, \frac{\mathbf{x}_{n-i}}{\|\mathbf{x}_0\|}\right) \frac{\alpha u^{-\alpha-1}}{\|\mathbf{x}_{-i}\|^{\alpha}} \mathbf{1}_{\{\|\mathbf{x}_{-i}\|>1,\, \|\mathbf{x}_0\|>u^{-1}\}} \nu(\mathbf{dx}) du \\ &= \int_{(\mathbb{R}^d)^{\mathbb{Z}}} f\left(\frac{\mathbf{x}_{m-i}}{\|\mathbf{x}_0\|}, \ldots, \frac{\mathbf{x}_{n-i}}{\|\mathbf{x}_0\|}\right) \frac{\|\mathbf{x}_0\|^{\alpha}}{\|\mathbf{x}_{-i}\|^{\alpha}} \mathbf{1}_{\{\|\mathbf{x}_{-i}\|>1\}} \nu(\mathbf{dx}) \\ &= \int_{(\mathbb{R}^d)^{\mathbb{Z}}} f\left(\frac{\mathbf{x}_{m}}{\|\mathbf{x}_i\|}, \ldots, \frac{\mathbf{x}_{n}}{\|\mathbf{x}_i\|}\right) \frac{\|\mathbf{x}_i\|^{\alpha}}{\|\mathbf{x}_{0}\|^{\alpha}} \mathbf{1}_{\{\|\mathbf{x}_{0}\|>1\}} \nu(\mathbf{dx}) \\ &= E\left[f\left(\frac{\boldsymbol{\Theta}_m}{\|\boldsymbol{\Theta}_i\|}, \ldots, \frac{\boldsymbol{\Theta}_n}{\|\boldsymbol{\Theta}_i\|}\right) \|\boldsymbol{\Theta}_i\|^{\alpha}\right]\,. \end{aligned}$$

Notice that a similar argument can prove statement (ii) of Theorem 3.1 in Basrak and Segers (2009) as well.

6.4 Application: Ergodic Theoretical Properties of Tail Measures and Those of Probability Laws of $(X_t,\, t \in T)$

In this section, we will always consider *a stationary* process $\mathbf{X} = (X_t,\, t \in T)$ with $T = \mathbb{Z}$ or $\mathbb{R}$, assuming that $\mathbf{X}$ has regularly varying tails. Let ν be the tail measure of $\mathbf{X}$. As pointed out in the preceding section, ν is shift invariant:

$$\nu \circ \phi_t^{-1} = \nu \quad \text{for every } t \in T\,,$$

where $\phi_t : \mathbb{R}^T \to \mathbb{R}^T$ is defined by $\phi_t(\mathbf{x}) = \mathbf{x}_{t+\cdot}$, $\mathbf{x} \in \mathbb{R}^T$. Thus, we are motivated to study the properties of the tail measure from ergodic-theoretical viewpoint. In particular, we will investigate the connection between the ergodic theoretical properties of the tail measure and those of the probability law of the process $\mathbf{X}$.

Here we need to recall the so-called positive-null decomposition by which the ergodic properties of the tail measure will be rigorously described. For details, we refer to Wang et al. (2011), which is essentially based on Takahashi (1971). See also Aaronson (1997) and Krengel (1985). First, suppose that a tail measure ν is σ-finite (a necessary and sufficient condition for σ-finiteness is given in Proposition 6.1.4). Then $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^T, \nu)$ becomes a standard Lebesgue measure space (see Appendix A in Pipiras and Taqqu (2004) for the terminology). We define

$$\Lambda = \left\{Q \ll \nu : Q \text{ is a finite measure on } \mathbb{R}^T,\ Q \circ \phi_t^{-1} = Q \text{ for all } t \in T\right\},$$

$$S_Q = \left\{\mathbf{x} \in \mathbb{R}^T : \frac{dQ}{d\nu}(\mathbf{x}) > 0\right\},\ \ Q \in \Lambda\,.$$

According to Lemma 2.2 in Wang et al. (2011), $\{S_Q : Q \in \Lambda\}$ has a unique maximal element P in the sense that

(i): for all $R \in \Lambda$, $\nu(S_R \setminus P) = 0$,

(ii): if there exists another P' satisfying (i), then $P = P' \mod \nu$.

Then, such P is called *a positive part* and $N = \mathbb{R}^T \setminus P$ is *a null part*. It is shown by

Theorem 2.3 in Wang et al. (2011) that both P and N are invariant with respect to $(\phi_t,\, t \in T)$, i.e., for all $t \in T$,

$$\mu\Big(\phi_t^{-1}(P) \triangle P\Big) = 0 \quad \text{and} \quad \mu\Big(\phi_t^{-1}(N) \triangle N\Big) = 0\,.$$

If $\mathbb{R}^T = P \mod \nu$, then $(\phi_t,\, t \in T)$ is said to be a positive flow, and if $\mathbb{R}^T = N \mod \nu$, then it is called a null flow.

Our first result below relates the ergodic properties of the flow $(\phi_t,\, t \in T)$ defined on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^T, \nu)$ to the Cesàro type convergence of $\nu\big\{\mathbf{x} \in \mathbb{R}^T : |x_0| > 1,\ |x_t| > 1\big\}$. More precisely, if $(\phi_t,\, t \in T)$ is a null flow, $\nu\big\{\mathbf{x} \in \mathbb{R}^T : |x_0| > 1,\ |x_t| > 1\big\}$ converges to zero in the Cesàro sense. On the contrary, if $(\phi_t,\, t \in T)$ has a positive component, then the same quantity does not converge to zero in the Cesàro sense. Alternatively, we may say that if $(\phi_t,\, t \in T)$ has a positive component, then the original process $\mathbf{X}$ exhibits stronger dependence among their extremes.

Proposition 6.4.1. *Let λ denote either counting measure (if $T = \mathbb{Z}$) or Lebesgue measure (if $T = \mathbb{R}$).*

(i): If $(\phi_t,\, t \in T)$ is a null flow on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^T, \nu)$, then

$$\frac{1}{T}\int_{[0,T]} \nu\big\{\mathbf{x} \in \mathbb{R}^T : |x_0| > \delta,\ |x_t| > \delta\big\}\lambda(dt) \to 0$$

for every $\delta > 0$.

(ii): If $(\phi_t,\, t \in T)$ has a positive component on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^T, \nu)$, then

$$\liminf_{T\to\infty} \frac{1}{T}\int_{[0,T]} \nu\big\{\mathbf{x} \in \mathbb{R}^T : |x_0| > \delta,\ |x_t| > \delta\big\}\lambda(dt) > 0$$

for every $\delta > 0$ with $\nu\big\{\mathbf{x} \in P : |x_0| > \delta\,\big\} > 0$. Here, P is a positive part of $\mathbb{R}^T$.

Proof. Because of an invariance property of ν, it suffices to check these statements when $\delta = 1$.

(i): Let $(\phi_t,\, t \in T)$ be a null flow defined on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^T, \nu)$. Then,

$$\frac{1}{T}\int_{[0,T]} \nu\big\{\mathbf{x} \in \mathbb{R}^T : |x_0| > 1,\ |x_t| > 1\big\}\lambda(dt)$$

$$= \int_0^1 \nu \left\{ \mathbf{x} \in A : \frac{1}{T} \int_{[0,T]} \mathbf{1}_A \circ \phi_t(\mathbf{x}) \lambda(dt) > y \right\} dy \,,$$

where $A = \{\mathbf{x} \in \mathbb{R}^T : |x_0| > 1\}$ is a measurable set of ν-finite measure. It follows from Krengel's stochastic ergodic theorem (see Theorem 4.9 of Krengel (1985)) that

$$\nu \left\{ \mathbf{x} \in A : \frac{1}{T} \int_{[0,T]} \mathbf{1}_A \circ \phi_t(\mathbf{x}) \lambda(dt) > y \right\} \to 0$$

for every $0 \le y \le 1$. So, the result follows.

(ii): By virtue of (i), we may assume without loss of generality that $(\phi_t,\, t \in T)$ is a positive flow on the whole measure space $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^T, \nu)$. Then, there exists a probability measure Q that is equivalent to ν and is preserved under $(\phi_t,\, t \in T)$. Let $g = dQ/d\nu$ be its Radon-Nikodym derivative. Let $A = \{\mathbf{x} \in \mathbb{R}^T : |x_0| > 1\}$. Since Q is a probability measure, the Birkhoff ergodic theorem yields

$$\frac{1}{T} \int_{[0,T]} \mathbf{1}_A \circ \phi_t(\mathbf{x}) \lambda(dt) \to E_Q\big(\mathbf{1}_A | \mathcal{I}\big), \quad Q\text{-a.e.},$$

where $\mathcal{I}$ is the σ-field of all $(\phi_t,\, t \in T)$-invariant measurable sets.

Consequently,

$$\frac{1}{T} \int_{[0,T]} \int_{A \cap \phi_t^{-1} A} g(\mathbf{x}) \nu(d\mathbf{x}) \lambda(dt)$$
$$= \int_A \frac{1}{T} \int_{[0,T]} \mathbf{1}_A \circ \phi_t(\mathbf{x}) \lambda(dt) Q(d\mathbf{x}) \to \int_A E_Q\big(\mathbf{1}_A | \mathcal{I}\big) dQ > 0 \,.$$

Choose $K > 0$ so that

$$\int_A g(\mathbf{x}) \mathbf{1}_{\{g(\mathbf{x}) > K\}} \nu(d\mathbf{x}) \le \frac{1}{2} \int_A E_Q\big(\mathbf{1}_A | \mathcal{I}\big) dQ \,.$$

Now, we have

$$\frac{1}{T} \int_{[0,T]} \int_{A \cap \phi_t^{-1} A} g(\mathbf{x}) \nu(d\mathbf{x}) \lambda(dt) \le \frac{1}{2} \int_A E_Q\big(\mathbf{1}_A | \mathcal{I}\big) dQ + \frac{K}{T} \int_{[0,T]} \nu\big(A \cap \phi_t^{-1} A\big) \lambda(dt) \,.$$

Therefore,

$$\liminf_{T \to \infty} \frac{1}{T} \int_{[0,T]} \nu\big(A \cap \phi_t^{-1} A\big) \lambda(dt) \ge \frac{1}{2K} \int_A E_Q\big(\mathbf{1}_A | \mathcal{I}\big) dQ > 0 \,.$$

$\square$

In the sequel, we only focus on the process studied in Examples 6.2.1 and 6.2.2: with $T = \mathbb{Z}$ or $\mathbb{R}$,

$$X_t = \int_E f_t(x) dM(x), \;\; t \in T\,. \tag{6.21}$$

Here M is an independently scattered infinitely divisible random measure on a measurable space $(E, \mathcal{E})$ with local Lévy measure $\rho(s, \cdot)$, $s \in E$, and σ-finite control measure m. Assume that M has no Gaussian component. In other words, the characteristic function of $M(A)$, for m-finite set $A \in \mathcal{E}$, is given by

$$Ee^{iuM(A)} = \exp\left[\int_A \left\{ iub(s) + \int_{\mathbb{R}} \big(e^{iux} - 1 - iu\tau(x)\big)\big)\rho(s, dx)\right\} m(ds)\right], \tag{6.22}$$

where $b : E \to \mathbb{R}$ and $\tau(x) = x/\max\{1, |x|\}$. The functions f_t are defined by

$$f_t(x) = f \circ \psi_t(x)\,, \;\; x \in E,\; t \in T\,, \tag{6.23}$$

where $\psi_t : E \to E$, $t \in T$, is a family of measurable maps, and $f : E \to \mathbb{R}$ is a measurable function. ψ_t and f are taken in such a way that the resulting process $\mathbf{X} = (X_t,\, t \in T)$ becomes a stationary and well-defined infinitely divisible process; see Rajput and Rosiński (1989).

Examples 6.2.1 and 6.2.2 both have proved that the tail measure of $\mathbf{X}$ is $(\rho_* \times m) \circ h^{-1}$, where $\rho_*(dx)$ is defined by either (6.12) or (6.16). As seen in Proposition 6.4.1, the Cesàro convergence of

$$(\rho_* \times m) \circ h^{-1}\big\{\mathbf{x} \in \mathbb{R}^T : |x_0| > \delta,\; |x_t| > \delta\,\big\}$$

is characterized by the ergodic theoretical properties of the flow $(\phi_t,\, t \in T)$ defined on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^T)$. Moreover, ergodicity of the probability law of $\mathbf{X}$ is characterized by the Cesàro convergence of the Lévy measure of $\mathbf{X}$. Namely, $\mathbf{X}$ is ergodic if and only if, for every $\eta > 0$,

$$\frac{1}{T}\int_{[0,T]} (\rho \times m) \circ h^{-1}\big\{\mathbf{x} \in \mathbb{R}^T : |x_0| > \eta,\; |x_t| > \eta\,\big\}\lambda(dt) \to 0 \;\;\text{as } T \to \infty\,.$$

See e.g. Rosiński and Żak (1997). Due to the similarity of the Lévy measure $(\rho \times m) \circ h^{-1}$ and the tail measure $(\rho_* \times m) \circ h^{-1}$, strong connection between the ergodic properties of $(\phi_t,\, t \in T)$ and ergodicity of $\mathbf{X}$ is expected to exist.

Theorem 6.4.2. *Let $(X_t,\, t \in T)$ be a stationary infinitely divisible process of the form (6.21), where M is an independently scattered infinitely divisible random measure given in (6.22), and f_t is defined in (6.23). We assume (6.9) and (6.11) and, furthermore, a regularly varying function $H : (0,\infty) \to (0,\infty)$ is bounded away from infinity on every compact interval. We assume a stronger version of (6.10): that is, for all $v > 0$, there exists a $K(v) > 0$, such that*

$$\sup_{u \ge v} \frac{\rho\big(s,(u,\infty)\big)}{H(u)} \le K(v) w_+(s) \quad \text{and} \quad \sup_{u \ge v} \frac{\rho\big(s,(-\infty,-u)\big)}{H(u)} \le K(v) w_-(s) \tag{6.24}$$

for all $s \in E$. We put an extra assumption on the lower bound of the quantities in (6.9): there exists $u_0 > 0$ and $L > 0$, such that

$$\frac{\rho\big(s,(u_0,\infty)\big)}{H(u_0)} \ge L w_+(s) \quad \text{and} \quad \frac{\rho\big(s,(-\infty,-u_0)\big)}{H(u_0)} \ge L w_-(s) \tag{6.25}$$

for all $s \in E$.

Applying the positive-null decomposition to the tail measure $\nu = (\rho_* \times m) \circ h^{-1}$, we have $\nu = \nu|_N + \nu|_P$. Then $(X_t,\, t \in T)$ is ergodic if and only if $\nu|_P$ is identically zero.

Proof. Recall that $(X_t,\, t \in T)$ is ergodic if and only if, for every $\eta > 0$,

$$\frac{1}{T}\int_{[0,T]} (\rho \times m)\big\{(x,s) : |xf(s)| > \eta,\ |xf_t(s)| > \eta\,\big\}\lambda(dt) \to 0 \quad \text{as } T \to \infty\,. \tag{6.26}$$

First, we will prove that (6.26) is equivalent to

$$\frac{1}{T}\int_{[0,T]}\int_{A_t^{(\epsilon)}} \big(w_+(s) + w_-(s)\big) m(ds)\lambda(dt) \to 0 \quad \text{as } T \to \infty \tag{6.27}$$

for every $\epsilon > 0$, where $A_t^{(\epsilon)} = \big\{s \in E : |f(s)| > \epsilon,\ |f_t(s)| > \epsilon\,\big\}$.

Assume that (6.26) holds for every $\eta > 0$. For any $\epsilon > 0$, let $\delta = \epsilon u_0$. Then

$$\begin{aligned}
&\frac{1}{T}\int_{[0,T]} (\rho \times m)\big\{(x,s) : |xf(s)| > \delta,\ |xf_t(s)| > \delta\,\big\}\lambda(dt)\\
\ge{}& \frac{1}{T}\int_{[0,T]} (\rho \times m)\big\{(x,s) : |x| > u_0,\ |f(s)| > \epsilon,\ |f_t(s)| > \epsilon\,\big\}\lambda(dt)\\
&= \frac{H(u_0)}{T}\int_{[0,T]}\int_{A_t^{(\epsilon)}} \frac{\rho\big(s,\{x : |x| > u_0\}\big)}{H(u_0)} m(ds)\lambda(dt)\\
&\ge \frac{LH(u_0)}{T}\int_{[0,T]}\int_{A_t^{(\epsilon)}} \big(w_+(s) + w_-(s)\big) m(ds)\lambda(dt)\,.
\end{aligned}$$

Here, the last inequality follows from (6.25) and, thus, (6.26) completes one direction of the assertion.

Conversely, assume that (6.27) holds for any $\epsilon > 0$. For every $\eta > 0$, we split the integral in (6.26) into three parts.

$$
\begin{aligned}
&\frac{1}{T}\int_{[0,T]} (\rho \times m)\big\{(x,s) : |xf(s)| > \eta,\ |xf_t(s)| > \eta\big\}\lambda(dt) \\
&= \frac{1}{T}\int_{[0,T]}\int_{|f(s)|\le\delta} \rho\Big(s, \big\{x : |xf(s)| > \eta,\ |xf_t(s)| > \eta\big\}\Big) m(ds)\lambda(dt) \\
&+ \frac{1}{T}\int_{[0,T]}\int_{|f(s)|>\delta,\ |f_t(s)|\le\epsilon} \rho\Big(s, \big\{x : |xf(s)| > \eta,\ |xf_t(s)| > \eta\big\}\Big) m(ds)\lambda(dt) \\
&+ \frac{1}{T}\int_{[0,T]}\int_{|f(s)|>\delta,\ |f_t(s)|>\epsilon} \rho\Big(s, \big\{x : |xf(s)| > \eta,\ |xf_t(s)| > \eta\big\}\Big) m(ds)\lambda(dt) \\
&= I_1 + I_2 + I_3\,.
\end{aligned}
$$

Notice that $(\rho \times m)\big\{(x,s) : |xf(s)| > \eta\big\} < \infty$, since the process $\mathbf{X}$ is well-defined. For the first term I_1, the stationarity of the process and the Cauchy-Schwarz inequality give the upper bound

$$
I_1 \le (\rho \times m)\big\{(x,s) : |xf(s)| > \eta,\ |f(s)| \le \delta\big\}^{1/2}(\rho \times m)\big\{(x,s) : |xf(s)| > \eta\big\}^{1/2}.
$$

The right hand side above converges to zero as $\delta \downarrow 0$, by the dominated convergence theorem. Next, we get

$$
I_2 \le (\rho \times m)\big\{(x,s) : |xf(s)| > \eta,\ |x| > \eta/\epsilon\big\},
$$

which goes to zero as $\epsilon \downarrow 0$ by the dominated convergence theorem.
Fix $\delta > 0$ and $\epsilon > 0$ so small that both I_1 and I_2 are sufficiently small. Applying the Cauchy-Schwarz inequality,

$$
\begin{aligned}
I_3 \le \bigg(\frac{1}{T}\int_{[0,T]}\int_{|f(s)|>\delta,\ |f_t(s)|>\epsilon} &\rho\Big(s, \big\{x : |xf(s)| > \eta\big\}\Big) m(ds)\lambda(dt)\bigg)^{1/2} \\
&\times(\rho \times m)\big\{(x,s) : |xf(s)| > \eta\big\}^{1/2}.
\end{aligned}
$$

Thus, it suffices to show that, for every $\epsilon > 0$,

$$
\frac{1}{T}\int_{[0,T]}\int_{A_t^{(\epsilon)}} \rho\Big(s, \big\{x : |xf(s)| > \eta\big\}\Big) m(ds)\lambda(dt) \to 0 \quad \text{as } T \to \infty\,.
$$

From (6.24), we have

$$\frac{1}{T}\int_{[0,T]}\int_{A_t^{(\epsilon)}}\rho\Big(s,\big\{x:|xf(s)|>\eta\big\}\Big)m(ds)\lambda(dt)$$

$$\leq \frac{K(\eta/\sup_{s\in E}|f(s)|)}{T}\int_{[0,T]}\int_{A_t^{(\epsilon)}}\big(w_+(s)+w_-(s)\big)H\big(\eta|f(s)|^{-1}\big)m(ds)\lambda(dt)\,.$$

Since H is bounded away from infinity on every compact interval, the Potter bounds yields, for some $C>0$,

$$H\big(\eta|f(s)|^{-1}\big)\leq C\left(\frac{\eta}{\sup_{s\in E}|f(s)|}\right)^{-\alpha/2}<\infty$$

for all $s\in E$. Therefore, (6.27) completes the other direction of the assertion.

Now we have checked that (6.26) and (6.27) are equivalent. Observe that even if one replaces ρ with ρ_* defined in (6.12), statements (6.26) and (6.27) are still equivalent. In fact, ρ_* satisfies (6.9), (6.24) and (6.25), if we set $H(u)=u^{-\alpha}$. In conclusion, (6.26) is equivalent to

$$\frac{1}{T}\int_{[0,T]}(\rho_*\times m)\big\{(x,s):|xf(s)|>\eta,\ |xf_t(s)|>\eta\big\}\lambda(dt)\to 0\quad\text{as } T\to\infty \tag{6.28}$$

for every $\eta>0$. However, we find from Proposition 6.4.1 that (6.28) holds if and only if $\nu|_P$ is identically zero. $\square$

We will, next, study the process given in Example 6.2.2.

Theorem 6.4.3. *Let $(X_t,\,t\in T)$ be a stationary infinitely divisible process of the form (6.21), where M is an independently scattered infinitely divisible random measure given in (6.22), and f_t is defined in (6.23). However, we let ρ be independent of $s\in E$. We assume tail balanced regularly varying condition (6.15). We will specify the integrability of f as follows: for every $t\in T$,*

$$\begin{cases}\int_E|f_t(s)|^{\alpha-\xi}\vee|f_t(s)|^2m(ds)<\infty & \textit{for some } 0<\xi<2-\alpha \ \textit{ if } 0<\alpha<2\,,\\ \int_E|f_t(s)|^{\alpha-\xi}\vee|f_t(s)|^{\alpha+\xi}m(ds)<\infty & \textit{for some } 0<\xi<\alpha \ \textit{ if } \alpha\geq 2\,.\end{cases}$$

Furthermore, if $0 < \alpha < 2$, the lower tail of ρ is assumed to satisfy

$$x^{p_0}\rho\big(y : |y| > x\big) \to 0 \ \text{ as } x \downarrow 0 \tag{6.29}$$

for some $p_0 \in (\alpha, 2)$.

Under this setup, $(X_t,\ t \in T)$ is ergodic if and only if $\nu|_P$ is identically zero.

Proof. We only prove that

$$\frac{1}{T}\int_{[0,T]}(\rho \times m)\big\{(x,s) : |xf(s)| > \eta,\ |xf_t(s)| > \eta\big\}\lambda(dt) \to 0\,, \quad \text{for every } \eta > 0\,, \tag{6.30}$$

is equivalent to

$$\frac{1}{T}\int_{[0,T]} m\big(A_t^{(\epsilon)}\big)\lambda(dt) \to 0 \quad \text{for every } \epsilon > 0\,, \tag{6.31}$$

where $A_t^{(\epsilon)} = \big\{s \in E : |f(s)| > \epsilon,\ |f_t(s)| > \epsilon\big\}$. Once the above equivalence is established, the rest of the argument is almost the same as that in Theorem 6.4.2.

First, we assume (6.30). For any $\epsilon > 0$,

$$\begin{aligned}
&\frac{1}{T}\int_{[0,T]}(\rho \times m)\big\{(x,s) : |xf(s)| > \epsilon,\ |xf_t(s)| > \epsilon\big\}\lambda(dt) \\
\geq\ &\frac{1}{T}\int_{[0,T]}(\rho \times m)\big\{(x,s) : |x| > 1,\ |f(s)| > \epsilon,\ |f_t(s)| > \epsilon\big\}\lambda(dt) \\
&= \frac{\rho\big\{x : |x| > 1\big\}}{T}\int_{[0,T]} m\big(A_t^{(\epsilon)}\big)\lambda(dt)\,.
\end{aligned}$$

Thus, $T^{-1}\int_{[0,T]} m\big(A_t^{(\epsilon)}\big)\lambda(dt) \to 0$ as $T \to \infty$.

Assume, conversely, that (6.31) holds. Once again, we need split the integral in (6.30) into three terms. For every $\eta > 0$,

$$\begin{aligned}
&\frac{1}{T}\int_{[0,T]}(\rho \times m)\big\{(x,s) : |xf(s)| > \eta,\ |xf_t(s)| > \eta\big\}\lambda(dt) \\
= &\frac{1}{T}\int_{[0,T]}\int_{|f(s)|\leq\delta}\rho\big(x : |xf(s)| > \eta,\ |xf_t(s)| > \eta\big)m(ds)\lambda(dt) \\
+ &\frac{1}{T}\int_{[0,T]}\int_{|f(s)|>\delta,\ |f_t(s)|\leq\epsilon}\rho\big(x : |xf(s)| > \eta,\ |xf_t(s)| > \eta\big)m(ds)\lambda(dt) \\
+ &\frac{1}{T}\int_{[0,T]}\int_{|f(s)|>\delta,\ |f_t(s)|>\epsilon}\rho\big(x : |xf(s)| > \eta,\ |xf_t(s)| > \eta\big)m(ds)\lambda(dt)
\end{aligned}$$

$$= I_1 + I_2 + I_3\,.$$

By a similar argument as the proof of Theorem 6.4.2, I_1 and I_2 can be arbitrarily small by taking $\delta > 0$ and $\epsilon > 0$ sufficiently small. Having fixed such $\delta > 0$ and $\epsilon > 0$ and assuming $\epsilon < \delta$ without loss of generality, we have

$$I_3 \le \frac{1}{T}\int_{[0,T]}\int_E \mathbf{1}_{A_t^{(\epsilon)}}(s)\rho\bigl(x : |xf(s)| > \eta\bigr)m(ds)\lambda(dt)\,.$$

If $0 < \alpha < 2$, an application of the Hölder's inequality provides

$$I_3 \le \left(\frac{1}{T}\int_{[0,T]} m\bigl(A_t^{(\epsilon)}\bigr)\lambda(dt)\right)^{1-p_0/2}\left(\frac{1}{T}\int_{[0,T]}\int_{A_t^{(\epsilon)}} \rho\bigl(x : |xf(s)| > \eta\bigr)^{2/p_0} m(ds)\lambda(dt)\right)^{p_0/2}.$$

By virtue of (6.31), it is enough to verify

$$\limsup_{T\to\infty}\frac{1}{T}\int_{[0,T]}\int_{A_t^{(\epsilon)}} \rho\bigl(x : |xf(s)| > \eta\bigr)^{2/p_0} m(ds)\lambda(dt) < \infty\,. \tag{6.32}$$

Indeed,

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{T}\int_{[0,T]}\int_{A_t^{(\epsilon)}} \rho\bigl(x : |xf(s)| > \eta\bigr)^{2/p_0} m(ds)\lambda(dt)\\ &\quad\le \int_E \rho\bigl(x : |x| > \eta|f(s)|^{-1}\bigr)^{2/p_0}\mathbf{1}_{\{\eta\le|f(s)|\}}m(ds)\,.\end{aligned}$$

Because of (6.29),

$$\rho\bigl(x : |x| > \eta|f(s)|^{-1}\bigr)\mathbf{1}_{\{\eta\le|f(s)|\}} \le C\bigl(\eta|f(s)|^{-1}\bigr)^{-p_0}$$

for some $C > 0$. Since $f \in L^2(E)$, (6.32) follows.

In case of $\alpha \ge 2$, let $\epsilon_0 \in (0,\xi)$. From the Hölder's inequality,

$$\begin{aligned}I_3 \le &\left(\frac{1}{T}\int_{[0,T]} m\bigl(A_t^{(\epsilon)}\bigr)\lambda(dt)\right)^{\epsilon_0/(\alpha+\xi)}\\ &\times\left(\frac{1}{T}\int_{[0,T]}\int_{A_t^{(\epsilon)}} \rho\bigl(x : |xf(s)| > \eta\bigr)^{(\alpha+\xi)/(\alpha+\xi-\epsilon_0)} m(ds)\lambda(dt)\right)^{1-\epsilon_0/(\alpha+\xi)}.\end{aligned}$$

In this case, we have, for some $C > 0$,

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{T}\int_{[0,T]}\int_{A_t^{(\epsilon)}} \rho\bigl(x : |xf(s)| > \eta\bigr)^{(\alpha+\xi)/(\alpha+\xi-\epsilon_0)} m(ds)\lambda(dt)\\ &\quad\le \int_E \rho\bigl(x : |x| > \eta|f(s)|^{-1}\bigr)^{(\alpha+\xi)/(\alpha+\xi-\epsilon_0)}\mathbf{1}_{\{\eta\le|f(s)|\}}m(ds)\end{aligned}$$

$$\leq C\eta^{-(\alpha+\xi)} \int_E |f(s)|^{\alpha+\xi} m(ds) < \infty\,.$$

The last inequality follows from $y^{\alpha+\xi-\epsilon_0}\rho\big(x : |x| > y\big) \to 0$ as $y \downarrow 0$. Now, in either case, $\limsup_{T\to\infty} I_3 = 0$ and, hence, (6.30) has been established. $\square$

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